Revision for The First Exam

Linear Algebra - Abdullah AlAzemi

1. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$. Compute $A^2 + I_2$.

2. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

   (a) Find A^{1977}.

   (b) Find all matrices B such that $AB = BA$.

3. Let $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. Find A^{100}.

4. Let $A = \begin{bmatrix} 1 & -2 \\ -3 & 0 \\ 2 & 6 \end{bmatrix}$, and $B = \begin{bmatrix} -2 & 3 & 4 \\ 3 & 2 & 1 \end{bmatrix}$. Find the columns of AB as a linear combination of columns of A.

5. Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & -3 \\ -3 & 1 & 4 \end{bmatrix}$. Express the third row of AB as a linear combination of the rows of B.

6. Let A be a 2×2 matrix and $B = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$. If $AB = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, find A.

7. Let A and B be two $n \times n$ matrices such that A is symmetric and B is skew-symmetric. Show that $AB + BA$ is a skew-symmetric matrix.

8. Let A be 2×2 skew-symmetric matrix. If $A^2 = A$, then $A = \mathbf{0}$.

9. If $A\,A^T = \mathbf{0}$, then $A = \mathbf{0}$.

10. Let $A = \begin{bmatrix} 1 \\ 3 \\ -1 \\ 3 \end{bmatrix}$. Find all constants $c \in \mathbb{R}^3$ such that $(cA)^T \cdot (cA) = 5$.

11. Let $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 1 \\ 1 & -2 & -3 \end{bmatrix}$. Find a symmetric matrix S and a skew symmetric matrix K such that $A = S + K$.

12. Find the reduced row echelon form (r.r.e.f.) of the following matrix:

$$\begin{bmatrix} 2 & 4 & 6 & 0 \\ 1 & 2 & 4 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}$$

13. Show that if a system $AX = B$ has more than one solution, then it has an infinite many solutions.

14. Discuss the consistency of the following system:

$$\begin{array}{rcrcrcl} x & & & + & z & = & 4 \\ 2x & + & y & + & 3z & = & 5 \\ -3x & - & 3y & + & (a^2-5a)z & = & a-8 \end{array}$$

15. Solve the following system using the Guass-Jordan method.

$$\begin{array}{rcrcrcrcrcl} x_1 & + & x_2 & + & x_3 & + & x_4 & + & 2x_5 & = & 0 \\ x_1 & & & & & + & x_4 & & & = & 0 \\ x_1 & + & 2x_2 & + & x_3 & & & - & x_5 & = & 0 \end{array}$$

16. Solve the following system:

$$\begin{array}{rcrcrcrcl} x & - & y & + & 2z & - & 2w & = & 1 \\ -x & + & 2y & - & z & & & = & -1 \\ 3x & & & - & 2z & - & w & = & -8 \\ & + & 6y & - & 5z & - & w & = & -11. \end{array}$$

17. Find all values of a for which the following system has:
(i) no solution, (ii) unique solution, (iii) infinite many solutions.

$$\begin{array}{rcrcrcl} x & + & 2y & - & z & = & 2 \\ 2x & + & 5y & + & z & = & 5 \\ x & + & y & + & (a^2-5)z & = & a \end{array}$$

18. Let $A = \begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 1 & -1 \\ 1 & -2 & 3 & 1 \\ 0 & 2 & 0 & a^2+1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ a+2 \end{bmatrix}$. Find all value(s) of a such that the system $A\mathbf{x} = B$ has at least one solution.

19. Solve the system

$$\begin{array}{rcrcrcl} \mathrm{x} & + & \mathrm{y} & + & \mathrm{z} & = & 0 \\ \mathrm{x} & + & 2\mathrm{y} & + & 3\mathrm{z} & = & 0 \\ \mathrm{x} & + & 3\mathrm{y} & + & 4\mathrm{z} & = & 0 \\ \mathrm{x} & + & 4\mathrm{y} & + & 5\mathrm{z} & = & 0. \end{array}$$

20. Consider the system:

$$\begin{array}{rcrcrcl} x & - & y & + & (a+3)z & = & a^3-a-7 \\ -x & + & ay & - & az & = & a \\ & & 2(a-1)y & + & (a^2+2)z & = & 8a-14. \end{array}$$

(a) Find all value(s) of a for which the system has:
(i) no solution, (ii) unique solution, and (iii) infinite many solutions.

(b) Solve this system for $a = 1$.

21. Show that if C_1 and C_2 are solutions of the system $A\mathbf{x} = B$, then $4C_1 - 3C_2$ is also a solution of this system.

22. Let $A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & -2 & 3 \\ 1 & 1 & -1 \end{bmatrix}$, and $B = \begin{bmatrix} 0 & 1 & -2 \\ 3 & 3 & -1 \\ -1 & 1 & -3 \end{bmatrix}$. Find a matrix C such that $AB^{-1}C = 2I_3$.

23. Let $AX = B$ be a linear system such that

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & -2 \\ -1 & -1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ and } B = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

(a) Find A^{-1},

(b) Use part (a) to solve $AX = B$.

24. Let $A^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$.

(a) Find A.

(b) Find C^T, if $CA^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 6 & 3 \end{bmatrix}$.

25. If $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -2 & 4 & 1 \end{bmatrix}$ and If $A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 3 & 6 & 1 \end{bmatrix}$, find A. [Hint: $A = A^3\,(A^2)^{-1}$.]

26. (a) Find A if $A^{-1} = \begin{bmatrix} 4 & -2 & 1 \\ 0 & -1 & 4 \\ 1 & -1 & 2 \end{bmatrix}$.

(b) Solve the linear system $A^{-1}X = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$.

27. Let $A^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Find C if $AC = B^T$.

[**Hint**: Consider multiplying both sides from the left with A^{-1}.]

28. Let $A^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. Find all $x, y, z \in \mathbb{R}^3$ such that $\begin{bmatrix} x & y & z \end{bmatrix} A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$.

29. Let $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & 0 \\ 1 & 1 & -1 \end{bmatrix}$.

(a) find B^{-1}.

(b) Find C if $A = BC$.

30. Let A, B, and C be $n \times n$ matrices such that $D = AB + AC$ is non-singular.

(a) Find A^{-1} is possible.

(b) Find $(B + C)^{-1}$ if possible.

31. Answer each of the following as a True or False. Justify your answer.

(a) If X_1, X_2 and X_3 are solutions of $AX = B$ $(B \neq 0)$, then $\frac{1}{2}X_1 + \frac{3}{2}X_2 - X_3$ is a solution to $AX = 0$.

(b) If A is a 2×2 skew-symmetric matrix, then $A^2 = cI_2$ where c is a real number.

(c) If A is a skew symmetric matrix, then AA^T is a skew symmetric matrix.

(d) For any non-singular (invertible) matrix A, $(A^{-1})^T = (A^T)^{-1}$.

(e) If A and B are non-singular $n \times n$ matrices, then $A + B$ is non-singular.

(f) If A and B are two $n \times n$ symmetric matrices, then AB is a symmetric matrix.

(g) If A and B are two $n \times n$ non-singular matrices, then $A^{-1} + B^{-1} = A^{-1}(A+B)B^{-1}$.

(h) If A is a 5×5 skew-symmetric matrix, then $|A| = 0$.

(i) If an $n \times n$ matrix A has inverse B, then B is unique.

(j) The system of linear equations $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 2 \\ 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ has a solution.