

# Revision for The First Exam

Linear Algebra - Abdullah AlAzemi

1. Let  $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ . Compute  $A^2 + I_2$ .
2. Let  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ .
  - (a) Find  $A^{1977}$ .
  - (b) Find all matrices  $B$  such that  $AB = BA$ .
3. Let  $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ . Find  $A^{100}$ .
4. Let  $A = \begin{bmatrix} 1 & -2 \\ -3 & 0 \\ 2 & 6 \end{bmatrix}$ , and  $B = \begin{bmatrix} -2 & 3 & 4 \\ 3 & 2 & 1 \end{bmatrix}$ . Find the columns of  $AB$  as a linear combination of columns of  $A$ .
5. Let  $A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 6 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & 1 & -3 \\ -3 & 1 & 4 \end{bmatrix}$ . Express the third row of  $AB$  as a linear combination of the rows of  $B$ .
6. Let  $A$  be a  $2 \times 2$  matrix and  $B = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ . If  $AB = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ , find  $A$ .
7. Let  $A$  and  $B$  be two  $n \times n$  matrices such that  $A$  is symmetric and  $B$  is skew-symmetric. Show that  $AB + BA$  is a skew-symmetric matrix.
8. Let  $A$  be  $2 \times 2$  skew-symmetric matrix. If  $A^2 = A$ , then  $A = \mathbf{0}$ .
9. If  $AA^T = \mathbf{0}$ , then  $A = \mathbf{0}$ .
10. Let  $A = \begin{bmatrix} 1 \\ 3 \\ -1 \\ 3 \end{bmatrix}$ . Find all constants  $c \in \mathbb{R}^3$  such that  $(cA)^T \cdot (cA) = 5$ .
11. Let  $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 1 \\ 1 & -2 & -3 \end{bmatrix}$ . Find a symmetric matrix  $S$  and a skew symmetric matrix  $K$  such that  $A = S + K$ .
12. Find the reduced row echelon form (r.r.e.f.) of the following matrix:
$$\begin{bmatrix} 2 & 4 & 6 & 0 \\ 1 & 2 & 4 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}$$
13. Show that if a system  $AX = B$  has more than one solution, then it has an infinite many solutions.

14. Discuss the consistency of the following system:

$$\begin{aligned} x &+ z = 4 \\ 2x + y + 3z &= 5 \\ -3x - 3y + (a^2 - 5a)z &= a - 8 \end{aligned}$$

15. Solve the following system using the Gauss-Jordan method.

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + 2x_5 &= 0 \\ x_1 &+ x_4 = 0 \\ x_1 + 2x_2 + x_3 &- x_5 = 0 \end{aligned}$$

16. Solve the following system:

$$\begin{aligned} x - y + 2z - 2w &= 1 \\ -x + 2y - z &= -1 \\ 3x &- 2z - w = -8 \\ &+ 6y - 5z - w = -11. \end{aligned}$$

17. Find all values of  $a$  for which the following system has:

(i) no solution, (ii) unique solution, (iii) infinite many solutions.

$$\begin{aligned} x + 2y - z &= 2 \\ 2x + 5y + z &= 5 \\ x + y + (a^2 - 5)z &= a \end{aligned}$$

18. Let  $A = \begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 1 & -1 \\ 1 & -2 & 3 & 1 \\ 0 & 2 & 0 & a^2 + 1 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ a + 2 \end{bmatrix}$ . Find all value(s) of  $a$  such that the system  $Ax = B$  has at least one solution.

19. Solve the system

$$\begin{aligned} x + y + z &= 0 \\ x + 2y + 3z &= 0 \\ x + 3y + 4z &= 0 \\ x + 4y + 5z &= 0. \end{aligned}$$

20. Consider the system:

$$\begin{aligned} x - y + (a + 3)z &= a^3 - a - 7 \\ -x + ay - az &= a \\ 2(a - 1)y + (a^2 + 2)z &= 8a - 14. \end{aligned}$$

(a) Find all value(s) of  $a$  for which the system has:

(i) no solution, (ii) unique solution, and (iii) infinite many solutions.

(b) Solve this system for  $a = 1$ .

21. Show that if  $C_1$  and  $C_2$  are solutions of the system  $Ax = B$ , then  $4C_1 - 3C_2$  is also a solution of this system.

22. Let  $A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & -2 & 3 \\ 1 & 1 & -1 \end{bmatrix}$ , and  $B = \begin{bmatrix} 0 & 1 & -2 \\ 3 & 3 & -1 \\ -1 & 1 & -3 \end{bmatrix}$ . Find a matrix  $C$  such that  $AB^{-1}C = 2I_3$ .

23. Let  $AX = B$  be a linear system such that

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & -2 \\ -1 & -1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ and } B = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

(a) Find  $A^{-1}$ ,

(b) Use part (a) to solve  $AX = B$ .

24. Let  $A^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$ .

(a) Find  $A$ .

(b) Find  $C^T$ , if  $CA^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 6 & 3 \end{bmatrix}$ .

25. If  $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -2 & 4 & 1 \end{bmatrix}$  and if  $A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 3 & 6 & 1 \end{bmatrix}$ , find  $A$ . [Hint:  $A = A^3 (A^2)^{-1}$ .]

26. (a) Find  $A$  if  $A^{-1} = \begin{bmatrix} 4 & -2 & 1 \\ 0 & -1 & 4 \\ 1 & -1 & 2 \end{bmatrix}$ .

(b) Solve the linear system  $A^{-1}X = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ .

27. Let  $A^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ . Find  $C$  if  $AC = B^T$ .

[Hint: Consider multiplying both sides from the left with  $A^{-1}$ .]

28. Let  $A^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ . Find all  $x, y, z \in \mathbb{R}^3$  such that  $[x \ y \ z]A = [1 \ 2 \ 3]$ .

29. Let  $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & 0 \\ 1 & 1 & -1 \end{bmatrix}$ .

(a) find  $B^{-1}$ .

(b) Find  $C$  if  $A = BC$ .

30. Let  $A, B$ , and  $C$  be  $n \times n$  matrices such that  $D = AB + AC$  is non-singular.

(a) Find  $A^{-1}$  is possible.

(b) Find  $(B + C)^{-1}$  if possible.

31. Answer each of the following as a True or False. Justify your answer.

(a) If  $X_1, X_2$  and  $X_3$  are solutions of  $AX = B$  ( $B \neq 0$ ), then  $\frac{1}{2}X_1 + \frac{3}{2}X_2 - X_3$  is a solution to  $AX = 0$ .

(b) If  $A$  is a  $2 \times 2$  skew-symmetric matrix, then  $A^2 = cI_2$  where  $c$  is a real number.

- (c) If  $A$  is a skew symmetric matrix, then  $AA^T$  is a skew symmetric matrix.
- (d) For any non-singular (invertible) matrix  $A$ ,  $(A^{-1})^T = (A^T)^{-1}$ .
- (e) If  $A$  and  $B$  are non-singular  $n \times n$  matrices, then  $A + B$  is non-singular.
- (f) If  $A$  and  $B$  are two  $n \times n$  symmetric matrices, then  $AB$  is a symmetric matrix.
- (g) If  $A$  and  $B$  are two  $n \times n$  non-singular matrices, then  $A^{-1} + B^{-1} = A^{-1}(A+B)B^{-1}$ .
- (h) If  $A$  is a  $5 \times 5$  skew-symmetric matrix, then  $|A| = 0$ .
- (i) If an  $n \times n$  matrix  $A$  has inverse  $B$ , then  $B$  is unique.
- (j) The system of linear equations 
$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 2 \\ 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$
 has a solution.