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HW # 4 in LINEAR ALGEBRA 111 / 02

Due Date is May 12, 2008

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1. Let $S = \{x_1, x_2, x_3, x_4\}$, where

$$x_1 = (1, 0, 1, 1), \qquad x_2 = (2, 0, 1, 1),$$
$$x_3 = (3, 2, 1, 0), \qquad x_4 = (0, 1, 2, 2).$$

   (a) Does S span $\mathbf{R}^4$? Justify your answer.

   (b) Is S a linearly dependent set or linearly independent set? Justify your answer.

   (c) Can S be considered as a basis for $\mathbf{R}^4$? Why?

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2. Find a basis for $\mathbf{R}^3$ that contains the vectors $x_1 = (1, 0, 3)$ and $x_2 = (2, 0, 3)$.

3. Find the dimension of the subspace W of $\mathbf{R}^3$, where W is the set of all vectors of the form $(a + b, b - c, a + c)$ with $a, b, c \in \mathbf{R}$.

4. Let $B_2 = \{x_1, x_2, \ldots, x_n\}$ be a basis for $\mathbf{R}^n$. Show that if A is an $n \times n$ non-singular matrix, then $B_2 = \{Ax_1, Ax_2, \ldots, Ax_n\}$ is a basis for $\mathbf{R}^n$.

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Review For LINEAR ALGEBRA 111

Answer Each of the following as True or False. Justify your answer.

1. (F) If U, V, and W are solutions of the linear system $AX = B$, then $4U - 3V + 2W$ is also a solution.
2. (F) If X and Y are two orthogonal vectors such that $||X|| = 3$ and $||Y|| = 4$, then $||Y - X|| = 1$.
3. (T) The set of all points on the plane $3x - 4y + 2z - 6 = 0$ does not form a subspace of $\mathbf{R}^3$.
4. (F) The vector $(-3, 2)$ belongs to span $\{(1, -2), (-2, 4)\}$.
5. (F) If A, B, and C are non-singular matrices with $B^T = AC$, then $B^{-1} = (C^{-1})^T (A^{-1})^T$.
6. (F) If X_1 and X_2 are solutions of the system $AX = B$ $(B \neq 0)$, then $\frac{1}{3}X_1 - \frac{4}{3}X_2$ is also a solution.
7. (F) The sign of the term $a_{12}a_{25}a_{33}a_{41}a_{54}$ in the expansion of a 5×5 determinant is $+$.
8. (T) If A is a 7×7 skew-symmetric matrix, then A is singular.
9. (F) The line $\frac{x-5}{4} = \frac{y-2}{-5} = \frac{z+3}{3}$ is perpendicular to the plane $4x + 5y + 3z = 9$.
10. (F) If A is a 6×4 matrix, then its rows are linearly independent.
11. (F) The angle between the two vectors $X = (1, 0, 0, 0)$ and $Y = (1, 0, 1, 1)$ is $\frac{\pi}{6}$.
12. (F) If X_1, X_2 and X_3 are solutions of $AX = B$ $(B \neq 0)$, then $\frac{1}{2}X_1 + \frac{3}{2}X_2 - X_3$ is a solution to $AX = 0$.
13. (F) The set of all 2×2 singular matrices is a subspace of $M_{2\times 2}$ (the space of all 2×2 matrices).
14. (T) If A is a 3×3 skew-symmetric matrix, then $|A| = 0$.
15. (F) If A and B are $n \times n$ matrices such that $A = B^T$, then $|2A^{-1}B| = 2$.
16. (T) If A and B are similar $n \times n$ matrices, then $|A| = |B|$.

Revision for Second Exam

1. Let A be a matrix with $A^{-1} = \begin{bmatrix} 7 & 1 & 0 & 3 \\ 2 & 0 & 0 & 0 \\ 1 & 3 & 5 & 4 \\ 6 & 2 & 0 & 5 \end{bmatrix}$. Find $\det(A)$ and $\text{adj}(A)$.

   [Hint: $\text{adj}(A) = |A|\, A^{-1}$].

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2. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \\ -1 & 3 & 1 \end{bmatrix}$.

   (a) Find $|A|$ using a cofactor expansion along row 2.

   (b) Find $Aadj(A)$.

   (c) Find $|2A^{-1}adj(A)|$.

3. Let A and B be two $n \times n$ matrices such that A is invertible and B is singular. Show that $A^{-1}B$ is singular.

4. Let A and B be two 4×4 non-singular matrices such that $|2A^{-1}B^2| = 32$ and $|B^{-1}adj(A)| = 4$. Find $|A|$ and $|B|$.

5. Let A be an $n \times n$ non-singular matrix. Show that if A is symmetric, then $adj(A)$ is symmetric.

6. Let $\boldsymbol{X} = (3, 2, -2)$ be a vector in $\mathbb{R}^3$. Compute $||\boldsymbol{X}||$ and then use the Cauchy-Schwartz Inequality to show that for any vector $\boldsymbol{Y} = (a, b, c)$, $(3a + 2b - 2c)^2 \leq 17(a^2 + b^2 + c^2)$.

7. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two vectors in $\mathbb{R}^n$. Show that $||\boldsymbol{X} - \boldsymbol{Y}|| \leq ||\boldsymbol{X}|| + ||\boldsymbol{Y}||$.

8. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two non-zeros vectors in $\mathbb{R}^3$, with angle $\theta = \pi/3$ between them. Find $||\boldsymbol{X} \times \boldsymbol{Y}||$, if $||\boldsymbol{X}|| = 3$ and $|| - 2\boldsymbol{Y}|| = 4$.

9. For any vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ in $\mathbb{R}^n$, show that $||\boldsymbol{X}|| \leq ||\boldsymbol{X} - 2\boldsymbol{Y}|| + 2||\boldsymbol{Y}||$.
   [Hint: use triangle inequality with $||\boldsymbol{X}|| = ||(\boldsymbol{X} - 2\boldsymbol{Y}) + 2\boldsymbol{Y}||$].

10. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two vectors in $\mathbb{R}^3$ such that $||\boldsymbol{X}|| = 2$ and $||\boldsymbol{Y}|| = 3$.

    (a) Find the maximum possible value for $||2\boldsymbol{X} + 3\boldsymbol{Y}||$.

    (b) If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are orthogonal, find $||2\boldsymbol{X} + 3\boldsymbol{Y}||$.

11. Let $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^n$. Find $\boldsymbol{X} \cdot \boldsymbol{Y}$ given that $||\boldsymbol{X} + \boldsymbol{Y}|| = 1$ and $||\boldsymbol{X} - \boldsymbol{Y}|| = 5$. [Hint: $||\boldsymbol{X} + \boldsymbol{Y}||^2 - ||\boldsymbol{X} - \boldsymbol{Y}||^2 = 4\boldsymbol{X} \cdot \boldsymbol{Y}$].

12. Let $\boldsymbol{L}$ be the line which is perpendicular (orthogonal) to the plane $\boldsymbol{\Pi} : 3x - y + 4z = 0$, and passes through the point $P(2, 6, 7)$. Find symmetric equations for $\boldsymbol{L}$.

13. Find parametric equations for the line $\boldsymbol{L}$ which passes through the points $P(2, -1, 4)$ and $Q(4, 4, -2)$. For what value of k is the point $R(k + 2, 14, -14)$ lie on the line $\boldsymbol{L}$.

14. Let $\boldsymbol{\Pi}_1 : 2x + 8y + 5z = 3$ and $\boldsymbol{\Pi}_2 : x + 4y + 2z = 1$ be two planes, and let $\boldsymbol{L}$ be the line of intersection of $\boldsymbol{\Pi}_1$ and $\boldsymbol{\Pi}_2$. Find parametric equations for $\boldsymbol{L}$.

15. Find an equation of the plane $\boldsymbol{\Pi}$ containing the lines: $\boldsymbol{L}_1$: $x = 3 + t$, $y = 1 - t$, $z = 3t$, and $\boldsymbol{L}_2$: $x = 2r$, $y = r - 2$, and $z = 5 - r$.

16. Find $a, b \in \mathbb{R}$ such that the point $P(3, a - 2b, 2a + b)$ lies on the line $\boldsymbol{L}$: $x = 1 + 2t$, $y = 2 - t$, and $z = 4 + 3t$. [Hint: try $t = 1$ and use the fact that P lies on $\boldsymbol{L}$].