[illegible]
Find the point Q on the plane $\mathbf{\Pi}$ such that the vector $\overrightarrow{PQ}$ is orthogonal to $\mathbf{\Pi}$.

18. Show that $\mathbb{W} = \{\, (a, b, c, d) \in \mathbb{R}^4 : 2b + c = d \text{ and } a - b = 0\}$ is a subspace of $\mathbb{R}^4$.

19. Let $\mathbb{V} = \{\, (x, y, z) : x, y, z \in \mathbb{R}\}$ and define $(x, y, z) \oplus (a, b, c) = (x + a, y + b, z + c)$ and $r \odot (x, y, z) = (rx, ry, 0)$. Show that $\mathbb{V}$ is not a vector space.

20. Determine whether $\mathbb{W}$ is a subspace of $\mathbb{R}^4$:

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    (a) $\mathbb{W} = \{(x, y, z, w) : x + y + z + w = 0\}$.
    (b) $\mathbb{W} = \{(a, b, c, d) : ad - bc = 0\}$.

21. Let $\boldsymbol{X} = (1, 2, 3, 4)$ and $\boldsymbol{Y} = (2, -1, -1, 0) \in \mathbb{R}^4$. Show that $\boldsymbol{Z} = (4, 3, 5, 8)$ belongs to $\text{Span}\{\boldsymbol{X}, \boldsymbol{Y}\}$.

22. Let $\mathbb{W} = \{(a, b, c) \,|\, 2a + b = c\}$. Show that $\mathbb{W}$ is a subspace of $\mathbb{R}^3$.

23. Consider the set $\mathbb{V} = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ with the operations $(x_1, y_1, z_1) \oplus (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$ and $c \odot (x, y, z) = (z, y, x)$. Is $\mathbb{V}$ a vector space? Explain.

24. Determine whether $\mathbb{W}$ is a subspace of $\mathbb{R}^3$:

    (a) $\mathbb{W}_1 = \{(a, b, c) : (a + b)c = 0\}$.
    (b) $\mathbb{W}_2$ is the set of all vectors in $\mathbb{R}^3$ which are orthogonal to $\boldsymbol{X} = (1, -1, 2)$.

25. Answer each question with "True" or "False" and justify your answer:

    (a) For a 5×5 matrix A, the sign of the term $a_{13}a_{21}a_{34}a_{45}a_{52}$ in $\det(A)$ is "+".
    (b) If A is a non-singular 3×3 matrix with $adj(A) = -A$, then $|A| = -1$.
    (c) There exist $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^4$ such that $||\boldsymbol{X}|| = ||\boldsymbol{Y}|| = 2$ and $\boldsymbol{X} \cdot \boldsymbol{Y} = 6$.
    (d) In $\mathbb{R}^3$, if $\boldsymbol{X} = 4\boldsymbol{Y}$, then $\boldsymbol{X} \times \boldsymbol{Y} = (0, 0, 0)$.
    (e) The vector $\boldsymbol{X} \times \boldsymbol{Y}$ is perpendicular to $2\boldsymbol{X} + 3\boldsymbol{Y}$.
    (f) $\mathbb{V} = \{(x, y) \in \mathbb{R}^2 : y < 0\}$ is closed under the operation $c \odot (x, y) = (cx, y)$.
    (g) If A and B are two 2×2 matrices such that $|adj(A)| = |adj(B)|$, then $A = B$.
    (h) In $\mathbb{R}^3$, the plane $\mathbf{\Pi} : y = 3$ is parallel to the y-axis.

6.1 Real Vector Spaces

Definition: A **real vector space** $\mathbb{V}$ is a set of elements with two operation $\oplus$ and $\odot$ satisfying the following conditions:

(α) if $x, y \in \mathbb{V}$, then $x \oplus y \in \mathbb{V}$, that is $\mathbb{V}$ is closed under $\oplus$.

- (a) for all $x, y \in \mathbb{V}$, we have $x \oplus y = y \oplus x$,
- (b) for all $x, y, z \in \mathbb{V}$, we have $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
- (c) for all $x \in \mathbb{V}$, there exists $O \in \mathbb{V}$ such that $x \oplus O = O \oplus x = x$.
- (d) for each $x \in \mathbb{V}$, there is $e \in \mathbb{V}$ such that $e \oplus x = x \oplus e = O$.

(β) if $x \in \mathbb{V}$ and $c \in \mathbb{R}$, then $c \odot x \in \mathbb{V}$, that is $\mathbb{V}$ is closed under $\odot$.

- (a) for all $x, y \in \mathbb{V}$, we have $c \odot (x \oplus y) = c \odot x \oplus c \odot y$, for all $c \in \mathbb{R}$,
- (b) for all $x \in \mathbb{V}$, we have $(c + d) \odot x = c \odot x \oplus d \odot x$ for all $c, d \in \mathbb{R}$,
- (c) for all $x \in \mathbb{V}$, we have $c \odot (d \odot x) = (cd) \odot x$ for all $c, d \in \mathbb{R}$,
- (d) for all $x \in \mathbb{V}$, we have $1 \odot x = x \odot 1 = x$.

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6.2 Subspaces

Definition: Let $(\mathbb{V}, \oplus, \odot)$ be a vector space, and let the non-empty set $\mathbb{W} \subseteq \mathbb{V}$. If $(\mathbb{W}, \oplus, \odot)$ is also a vector space, then we say that $\mathbb{W}$ is a **subspace** of $\mathbb{V}$.

Theorem: Let $(\mathbb{V}, \oplus, \odot)$ be a vector space, and let $\mathbb{W} \subseteq \mathbb{V}$. Then, $\mathbb{W}$ is a subspace of $\mathbb{V}$ if and only if the following conditions hold:

1. $\mathbb{W} \neq \emptyset$,
2. for all $x, y \in \mathbb{W}$, we have $x \oplus y \in \mathbb{W}$,
3. for all $x \in \mathbb{W}$ and $c \in \mathbb{R}$, we have $c \odot x \in \mathbb{W}$.

Definition: Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_n$ be vectors in a vector space $\mathbb{V}$. A vector $\boldsymbol{X} \in \mathbb{V}$ is called a **linear combination** of the vectors $\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_n$ if and only if

$$\boldsymbol{X} = c_1\boldsymbol{X}_1 + c_2\boldsymbol{X}_2 + \cdots + c_n\boldsymbol{X}_n,$$

for some real numbers $c_1, c_2, \cdots, c_n$.

Definition: Let $S = \{\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_k\}$ be a subset of a vector space $\mathbb{V}$. Then, the set of all vectors in $\mathbb{V}$ that are linear combination of the vectors of S is denoted by **span** S or **span** $\{\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_k\}$.

Theorem: Let $S = \{\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_k\}$ be a set of vectors of a vector space $\mathbb{V}$, then span S is a subspace of $\mathbb{V}$.

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HW # 5 in LINEAR ALGEBRA 111 / 02

Due Date is May 21, 2008

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1. Let $A = \begin{bmatrix} 1 & 1 & 1 & 2 & 3 \\ 4 & 6 & 0 & 0 & 0 \\ 3 & 5 & 3 & 6 & 9 \\ 2 & 3 & 2 & 4 & 6 \end{bmatrix}$.

(a) Find a basis for the row space of A that contains only rows of A.

(b) Find a basis for the column space of A.

(c) What is the nullity of A.

2. Let $A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 2 & 4 & 2 & 4 \end{bmatrix}$.

(a) Find bases for the row space and the column space of A.

(b) Find a basis for the null space of A (i.e. the solution space of the system $Ax = 0$).

(c) What is the nullity and the rank of A.

3. Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & 4 \\ 0 & -4 & 5 \end{bmatrix}$. Determine whether the matrix A is diagonalizable. Justify your answer.

4. Let $A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

(a) Find the eigenvalues of A.

(b) Find the corresponding eigenvectors of A.

(c) If possible, find a non-singular matrix P and a diagonal matrix D such that $D = P^{-1}AP$.