Kuwait University | *Final Exam Math 363* | *Date: December 19, 2015*
Department of Mathematics | *Dr. Latif A-M. Hanna* | *Duration: Two Hours*

Answer the following questions

1. Let F be a field, $\mathsf{V} = \mathsf{M}_{n \times n}(F)$ and W be the subset of V of all symmetric matrices of trace zero.

   (a) Show that W is a subspace of V.

   (b) Give a basis for W and determine $\dim \mathsf{W}$. (3+3 pts.)

2. Let $\mathsf{T} : \mathsf{P}_2(\mathbb{R}) \longrightarrow \mathsf{M}_{2\times 2}(\mathbb{R})$ be linear, defined by:
$$\mathsf{T}(a + bx + cx^2) = \begin{bmatrix} a+c & 2a+b+3c \\ b+c & a+b+2c \end{bmatrix}.$$

   (a) Find a basis for $\mathsf{N}(\mathsf{T})$.

   (b) Find a basis for $\mathsf{R}(\mathsf{T})$. (3+3 pts.)

3. Let T be the linear operator on $\mathsf{P}_1(\mathbb{R})$ defined by $\mathsf{T}(ax+b) = (3a-b)x + (a+3b)$ and let $\beta = \{x, 2\}$ and $\beta' = \{2+4x, 6-4x\}$ be ordered bases for $\mathsf{P}_1(\mathbb{R})$.

   (a) Find $[\mathsf{T}]_\beta$.

   (b) Find the change of coordinate matrix that changes β'-coordinates into β-coordinates and use it to find $[\mathsf{T}]_{\beta'}$. (2+4 pts.)

4. Let $\mathbf{f} : \mathbb{R}^3 \longrightarrow F$, be a linear functional on $\mathbb{R}^3$, defined by $\mathbf{f}(x,y,z) = x + 2y - 3z$. Let $\beta = \{(0,1,0), (1,0,0), (0,0,-1)\}$ be an ordered basis for $\mathsf{V} = \mathbb{R}^3$.

   (a) Find the dual basis β^* for V^*.

   (b) Express $\mathbf{f}$ as a linear combination of the elements of β^*. (3+3 pts.)

5. Let $S = \{(1,0,1), (0,0,-1), (0,1,1)\}$ be a basis for $\mathbb{R}^3$ and $x = (3,-7,4) \in \mathbb{R}^3$.

   (a) Apply Gram-Schmidt process to S to obtain an orthogonal basis, β, for $\mathbb{R}^3$.

   (b) Normalize the vectors in β, to obtain an orthonormal basis, β' for $\mathbb{R}^3$.

   (c) Use part (b), to get $[x]_{\beta'}$. (2+2+2 pts.)

6. Complete the following: (1 pt. each)

   (a) $S = \left\{ \begin{pmatrix} 1 & -3 & 2 \\ 0 & 0 & 5 \end{pmatrix}, \begin{pmatrix} -3 & 0 & 4 \\ 0 & -2 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 0 & 11 \\ -1 & 0 & 2 \end{pmatrix} \right\} \subseteq M_{2\times 3}(\mathbb{R})$, is linearly ... .

   (b) $\langle (1+i, 1-2i), (1-i, 2i) \rangle = ...$ in $\mathbb{C}^2$.

   (c) Let $T : \mathbb{C}^3 \longrightarrow \mathbb{R}^4$ be linear T is $1-1$ iff ... $(T) =$ ... and T is onto iff ... $(T) =$ .... (Hint: complete with rank, nullity or some integer numbers)

   (d) Let W be a subspace of $P_4(\mathbb{Z}_7)$. If $\dim(W) = 2$, then $\dim(W^\perp) =$ ... .

P.T.O.

7. Label the following statements as true or false. (1 pt. each)

(a) Let β be an ordered basis for a vector space V. If $u, v \in \beta$ and $u \neq v$, then $c_1 u + c_2 v = 0$ for some scalars $c_1 \neq c_2$.

(b) $\mathbb{C}$ is a vector space over $\mathbb{C}$ of dimension 2.

(c) $\langle A, B\rangle = tr\,(A + B)$ is an inner product on $M_{2\times 2}(\mathbb{R})$.

(d) $\langle f(x), g(x)\rangle = f(0)\, g(0)$ is an inner product on $P_2(\mathbb{C})$.

8. Answer only <u>*three*</u> of the following: (4 pts. each)

(a) Let W be a subspace of a vector space V. If O_W and O_V are the zero vectors of W and V, respectively. Show that $O_W = O_V$.

(b) Prove that the span of any subset S of a vector space V is a subspace of V. And show that any subspace of V that contains S must also contain the span of S.

(c) Let V and W be vector spaces over $\mathbb{Q}$. Prove that if $\mathsf{T} : V \longrightarrow W$ such that $\mathsf{T}(x + y) = \mathsf{T}(x) + \mathsf{T}(y)$, then T is linear.

(d) Let V and W be vector spaces over a field F, and let $\mathsf{T} : V \longrightarrow W$ be linear. If K is a subspace of V and $\mathsf{T}(K) = \{\mathsf{T}(x) : x \in K\}$, then prove that $\mathsf{T}(K)$ is a subspace of W.

(e) Let V be an inner product space over F. Prove that for $x, y, z \in V$ and $c \in F$:

1. $\langle 0, x\rangle = \langle x, 0\rangle = 0$.
2. $\|x\| = 0$ if and only if $x = 0$.
3. If $\langle x, y\rangle = \langle x, z\rangle$ for all $x \in V$, then $y = z$.

(f) Let $S = \{x_1, x_2, ..., x_k\}$ be an orthogonal set of nonzero vectors in an inner product space V. Prove that S is linearly independent.

(g) Prove that if W is a finite-dimentional subspace of an inner product space V, then $\left(W^{\perp}\right)^{\perp} = W$.

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1. (a) $\mathsf{tr}(O_{n\times n}) = 0$ and $O^t_{n\times n} = O_{n\times n}$, then $O_{n\times n} \in \mathsf{W}$. Let $A, B \in \mathsf{W}$, then $A^t = A, B^t = B$ and $(cA)^t = cA\ \forall c \in F$, $(A+B)^t = A+B$ and also, $\mathsf{tr}(cA) = 0 = \mathsf{tr}(A+B)$. Therefore, $cA, A+B \in \mathsf{W}$.

   (b) Basis for W
   $= \{E^{11} - E^{22}, E^{11} - E^{33}, ..., E^{11} - E^{nn}\} \cup \{E^{ij} + E^{ji} : i \neq j, i, j = 1, 2, ..., n\}$.
   $\dim \mathsf{W} = (n-1) + \frac{n^2-n}{2} = \boxed{\frac{n^2+n-2}{2} = \frac{(n+2)(n-1)}{2}}$

2. (a) To find $\mathsf{N}(\mathsf{T})$, solve the linear system: $\begin{matrix} a + c = 0 & 2a + b + 3c = 0 \\ b + c = 0 & a + b + 2c = 0 \end{matrix}$.
   $a = -k, b = -k, c = k; k \in \mathbb{R}$. Thus, $\mathsf{N}(\mathsf{T}) = \{-k = kx + kx^2 : k \in \mathbb{R}\}$.
   Basis for $\mathsf{N}(\mathsf{T}) = \{1 + x - x^2\}$.

   (b) $\mathsf{R}(\mathsf{T}) = span\left(\mathsf{T}(1) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \mathsf{T}(x) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \mathsf{T}(x^2) = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}\right)$.
   But $\begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.
   Thefore, Basis for $\mathsf{R}(\mathsf{T}) = \left\{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right\}$.

3. (a) $\mathsf{T}(x) = 3x + 1 = (3)(x) + \left(\frac{1}{2}\right)(2), \mathsf{T}(2) = -2x + 6 = (-2)(x) + (3)(2)$. Thus, $[\mathsf{T}]_\beta = [\mathsf{T}]^\beta_\beta = \begin{bmatrix} 3 & -2 \\ \frac{1}{2} & 3 \end{bmatrix}$.

   (b) $2 + 4x = (4)(x) + (1)(2), 6 - 4x = (-4)(x) + (3)(2)$. Then $Q = \{I_V\}^\beta_{\beta'} = \begin{bmatrix} 4 & -4 \\ 1 & 3 \end{bmatrix}, Q^{-1} = \frac{1}{16}\begin{bmatrix} 3 & 4 \\ -1 & 4 \end{bmatrix}$. Since, $[\mathsf{T}]_{\beta'} = Q^{-1}[\mathsf{T}]_\beta Q$, then
   $$[\mathsf{T}]_{\beta'} = \frac{1}{16}\begin{bmatrix} 3 & 4 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ \frac{1}{2} & 3 \end{bmatrix}\begin{bmatrix} 4 & -4 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{25}{8} & -\frac{13}{8} \\ \frac{5}{8} & \frac{23}{8} \end{bmatrix}.$$

4. (a) Let $\beta = \{(0,1,0), (1,0,0), (0,0,-1)\} = \{x_1, x_2, x_3\}$ and $\beta^* = \{f_1, f_2, f_3\}$ where
   $f_1(x_1) = f_1(e_2) = 1, f_1(x_2) = f_1(e_1) = 0, f_1(x_3) = -f_1(e_3) = 0 \implies f_1(x,y,z) = y$
   $f_2(x_1) = f_2(e_2) = 0, f_2(x_2) = f_2(e_1) = 1, f_2(x_3) = -f_2(e_3) = 0 \implies f_2(x,y,z) = x$
   $f_3(x_1) = f_3(e_2) = 0, f_3(x_2) = f_3(e_1) = 0, f_3(x_3) = -f_3(e_3) = 1 \implies f_3(x,y,z) = -z$

   (b) Since, $\mathbf{f}(x_1) = 2, \mathbf{f}(x_2) = 1, \mathbf{f}(x_3) = 3$, then $\mathbf{f} = 2f_1 + f_2 + 3f_3$.

5. (a) Let $\beta = \{v_1, v_2, v_3\}, v_1 = w_1 = (1,0,1), v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2}v_1 = (0,0,-1) - \frac{\langle (0,0,-1),(1,0,1)\rangle}{\|(1,0,1)\|^2}(1,0,1) = (0,0,-1) - \left(-\frac{1}{2}\right)(1,0,1) = \left(\frac{1}{2}, 0, -\frac{1}{2}\right)$,
   $v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{\|v_1\|^2}v_1 - \frac{\langle w_3, v_2\rangle}{\|v_2\|^2}v_2$
   $= (0,1,1) - \frac{\langle (0,1,1),(1,0,1)\rangle}{\|(1,0,1)\|^2}(1,0,1) - \frac{\langle (0,1,1),\left(\frac{1}{2},0,-\frac{1}{2}\right)\rangle}{\|\left(\frac{1}{2},0,-\frac{1}{2}\right)\|^2}\left(\frac{1}{2},0,-\frac{1}{2}\right) = (0,1,1) - \left(\frac{1}{2}\right)(1,0,1) - (-1)\left(\frac{1}{2},0,-\frac{1}{2}\right) = (0,1,0)$.

Therefore, $\beta = \left\{(1,0,1), \left(\frac{1}{2}, 0, -\frac{1}{2}\right), (0,1,0)\right\}$.

(b) $\beta' = \left\{\frac{1}{\sqrt{2}}(1,0,1), \sqrt{2}\left(\frac{1}{2}, 0, -\frac{1}{2}\right), (0,1,0)\right\} = \left\{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\right), (0,1,0)\right\}$.

(c) $\langle x, v_1\rangle = \frac{7}{\sqrt{2}}, \langle x, v_2\rangle = -\frac{1}{\sqrt{2}}, \langle x, v_3\rangle = -7 \implies x = \left(\frac{7}{\sqrt{2}}\right)v_1 + \left(-\frac{1}{\sqrt{2}}\right)v_2 + (-7)\,v_3$.

6. (a) independent.

(b) $\langle (1+i, 1-2i), (1-i, 2i)\rangle = -4$

(c) $\text{nullity}(T) = 0, \text{rank}(T) = 4$.

(d) $\dim\left(W^{\perp}\right) = 5 - 2 = 3$

7. (a) F (b) F (c) F (d) F.