Kuwait University	Math 352	Date: May 28, 2017
Dr. Michael Johnson	Final Exam	Duration: 120 minutes

Solve exactly 7 of the following 8 questions. Each question counts 12 points.

1. Let $\mathcal{F}$ be the floating point system of numbers $\pm a.bc \times 10^d$, where $a, b, c \in \{0, 1, 2, \ldots, 9\}$, $a \neq 0$, and $d \in \{-2, -1, 0, 1, 2\}$. For $x \in \mathbb{R}$, let $fl(x)$ denote the closest element of $\mathcal{F}$ to x.
Let $x_1, x_2 \in \mathbb{R}$ be such that $fl(x_1) = 3.14$ and $fl(x_2) = 2.95$, and note that x_1 necessarily belongs to the interval $[a_1, b_1] := [3.135, 3.145]$. Moreover, $\mathcal{F} \cap [a_1, b_1] = 3.14$.

4 (a) Find the smallest interval $[a_2, b_2]$ which necessarily contains x_2, and verify that $\mathcal{F} \cap [a_2, b_2] = \{2.95\}$.

4 (b) Show that $x_1 - x_2$ necessarily belongs to the interval $[c, d] := [a_1 - b_2, b_1 - a_2]$.

4 (c) Find $\mathcal{F} \cap [c, d]$.

2. Define $g(x) = x - \frac{1}{6}(x^2 - 9)$, and let $\{x_n\}$ be defined recursively by $x_0 = 2$ and $x_n = g(x_{n-1})$ for $n = 1, 2, 3, \ldots$.

4 (a) Show that g is contractive on $[2, 5]$.

4 (b) Show that $g([2, 5]) \subset [2, 5]$.

4 (c) Show that $\{x_n\}$ converges quadratically.

3. Let $f(x) = \sqrt{x + 4}$, $-1 \leq x \leq 1$.

6 (a) Use Newton's divided differences to find the polynomial $p \in \Pi_2$ that interpolates f at the Chebyshev nodes.

6 (b) Show that $|f(x) - s(x)| \leq 0.0010024$ for all $x \in [-1, 1]$.

4. Find the function $s \in C^2[-1, 1]$ that minimizes $\displaystyle\int_{-1}^{1} [s''(x)]^2 \, dx$

~~8~~ 12 subject to $s(-1) = 0$, $s'(-1) = 2$, $s(0) = 1$, $s(1) = 0$, $s'(1) = -2$.

5. Let $V = C[0, 2]$ with the usual inner product, and let $W = \text{span}(w_1, w_2, w_3)$, where $w_1(x) = 1$, $w_2(x) = x - 1$, and $w_3(x) = 3x^2 - 6x + 2$.

4 (a) Show that w_1, w_2, w_3 form an orthogonal basis for W.

8 (b) Suppose $f \in V$ satisfies $\int_0^2 f(x)\, dx = 2$, $\int_0^2 x f(x)\, dx = -1$, and $\int_0^2 x^2 f(x)\, dx = 1$. Given that $(w_3, w_3) = 8/5$, find the function $w \in W$ that is closest to f.

6. Let W be a finite dimensional subspace of an inner product space V, and let $v_1, v_2 \in V$. Let w_1 be the orthogonal projection of v_1 onto W, and let w_2 be the orthogonal projection of v_2 onto W. Show that $2w_1 + 3w_2$ is the orthogonal projection $2v_1 + 3v_2$ onto W.

7. Consider the differentiation rule $D(f) = \dfrac{1}{6}[Af(0) + 18f(1) - 9f(2) + 2f(3)]$.

6 (a) Given that there exists a constant A such that $D(p) = p'(0)$ for all $p \in \Pi_3$, find A.

6 (b) Find the constant B such that, for all $f \in C^4[0, 3]$,

$$f'(0) = \frac{1}{6}[Af(0) + 18f(1) - 9f(2) + 2f(3)] + Bf^{(4)}(\xi) \text{ for some } \xi \in [0, 3],$$

8. Let $I(f) := \displaystyle\int_{-1}^{1} f(x)\, dx$ and consider the quadrature rule $\widetilde{I}(f) := Af(-1/\sqrt{3}) + Bf(1/\sqrt{3})$.

6 (a) Find A and B such that $\widetilde{I}(1) = I(1)$ and $\widetilde{I}(x) = I(x)$.

6 (b) Show that $\widetilde{I}(p) = I(p)$ for all $p \in \Pi_3$.