Kuwait University
Department of Mathematics

Differential Equations (Math 240) — 6 August 2017
Final Exam — Duration 2 hours

Answer all questions. Calculators and mobile phones are not allowed.

Q1. [10 pts.] Solve the following differential equation

$$(x+2y+1)dx+(2x+4y+3)dy=0. \tag{1}$$

Q2. [10 pts.] Use the variation of parameters method to solve the following initial value problem

$$y''-4y'+3y=\frac{e^{-x}}{1+e^{-4x}}. \tag{2}$$

Q3. [10 pts.] Use the method of undetermined coefficients to find the general solution of the following differential equation:

$$y'''-y''+y'-y=2e^{t}. \tag{3}$$

Q4. [10 pts.] Obtain the first five nonzero terms in a power series solution about $x=1$ for the initial value problem:

$$\frac{d^2y}{dx^2}+(x-1)y=1, \qquad y(1)=1,\ y'(1)=2. \tag{4}$$

Q5. [10 pts.] Find $L^{-1}\left\{\dfrac{s+3}{s^2+3s+2}\right\}$.

Q6. [20 pts.] Find the Laplace transform of the following functions:

a) $\displaystyle\int_0^t \beta^3 e^{3(t-\beta)}d\beta.$

b) $te^{2t}\sin t.$

Q7. [10+10=20 pts.]

a) Show that if $f(t)$ is continuous for $t\geq 0$ and of exponential order as $t\to\infty$, and if $f'(t)$ is piecewise continuous and of exponential order as $t\to\infty$, then:

$$L\{f'(t)\}=sL\{f(t)\}-f(0).$$

b) Use the Laplace transform method to solve the initial value problem:

$$y''+y=\cos 2t,\ \ y(0)=0,\ \ y'(0)=1. \tag{5}$$

Q8. [3+7=10 pts.] Consider the following initial value problem:

$$y''(t)+y(t)=H(t),\ y(0)=0,\ y'(0)=1 \tag{6}$$

where

$$H(t)=\begin{cases}-1, & 0\leq t\leq 3\\ 5-2t & t>3.\end{cases}$$

a) Write $H(t)$ in terms of the unit step function.

b) Use part (a) to solve the initial value problem.

Q1. [10 pts.]

Let $u = x + 2y$. Then, $du = dx + 2dy \implies dx = du - 2dy$. Substituting into the equation and simplifying results in: $(u + 1)du + dy = 0$. Integrating, we get: $\frac{1}{2}u^2 + u + y = c, \implies \frac{1}{2}(x + 2y)^2 + (x + 2y) + y = c$.

Q2. [10 pts.]

$y'' - 4y' + 3y = 0, \implies y_c = c_1 e^x + c_2 e^{3x}$. Using Variation of parameters: $y_p = A(x)e^x + B(x)e^{3x}$:

$$\begin{cases} A'(x)e^x + B'(x)e^{3x} &= 0 \\ A'(x)e^x + 3B'(x)e^{3x} &= \dfrac{e^{-x}}{1 + e^{-4x}} \end{cases}$$

Solving the above system for $A'(x)$ and $B'(x)$, we get: $A'(x) = -\dfrac{e^{-2x}}{2(1 + e^{-4x})} \implies A(x) = \dfrac{1}{4}\tan^{-1}(e^{-2x})$, and $B'(x) = \dfrac{e^{-4x}}{2(1 + e^{-4x})} \implies B(x) = -\dfrac{1}{8}\ln(1+e^{-4x})$. So, $y_p = \dfrac{1}{4}e^x \tan^{-1}(e^{-2x}) - \dfrac{1}{8}e^{3x}\ln(1+e^{-4x})$. Thus, the general solution is $y = c_1e^x + c_2e^{3x} + \dfrac{1}{4}e^x \tan^{-1}(e^{-2x}) - \dfrac{1}{8}e^{3x}\ln(1+e^{-4x})$.

Q3. [10 pts.]
The auxiliary equation is: $r^3 - r^2 + r - 1 = (r - 1)(r^2 + 1) = 0 \implies r = 1, \pm i$. So $y_c(t) = c_1e^t + c_2 \cos t + c_3 \sin t$. The particular solution is $y_p(t) = Ate^t$, $y_p' = A(e^t + te^t)$, $y_p'' = A(2e^t + te^t)$, $y_p''' = A(3e^t + te^t)$. Substitution gives $A = 1$. The general solution is $y = c_1e^t + c_2 \cos t + c_3 \sin t + te^t$.

Q4. [10 pts.]
Let $y = \sum_{n=0}^{\infty} a_n(x - 1)^n$. Then $y' = \sum_{n=1}^{\infty} a_n n(x - 1)^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n - 1)a_n(x - 1)^{n-2}$. Substituting into the equation gives: $\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + (x-1)\sum_{n=0}^{\infty} a_n(x-1)^n = 1$ which is equivalent to: $\sum_{n=0}^{\infty}(n + 2)(n + 1)a_{n+2}(x - 1)^n + \sum_{n=0}^{\infty} a_n(x - 1)^{n+1} = 1 \iff$
$2a_2 + \sum_{n=1}^{\infty}(n + 2)(n + 1)a_{n+2}(x - 1)^n + \sum_{n=1}^{\infty} a_{n-1}(x - 1)^n = 1$ which leads to $2a_2 = 1$, &
$a_{n+2} = -\dfrac{a_{n-1}}{(n + 2)(n + 1)}$ for $n \geq 1$. Thus, $a_0 = y(1) = 1$, $a_1 = y'(1) = 2$, $a_2 = \dfrac{1}{2}$, $a_3 = -\dfrac{1}{6}a_0 = -\dfrac{1}{6}$; $a_4 = -\dfrac{1}{12}a_1 = -\dfrac{1}{6}$. The solution is: $y = 1 + 2(x-1) + \frac{1}{2}(x-1)^2 - \dfrac{1}{6}(x-1)^3 - \dfrac{1}{6}(x-1)^4 + \cdots$.

Q5. [10 pts.]
$\dfrac{s + 3}{s^2 + 3s + 2} = \dfrac{A}{s + 1} + \dfrac{B}{s + 2} = \dfrac{2}{s + 1} - \dfrac{1}{s + 2}$. Thus, $L^{-1}\left\{\dfrac{s + 3}{s^2 + 3s + 2}\right\} = L^{-1}\left\{\dfrac{2}{s + 1}\right\} - L^{-1}\left\{\dfrac{1}{s + 2}\right\} = 2e^{-t} - e^{-2t}$.

Q6. [20 pts.]
a) $L\left\{\int_0^t \beta^3 e^{3(t-\beta)}d\beta\right\} = L\{t^3\}\, L\{e^{3t}\} = \dfrac{6}{s^4}\dfrac{1}{s - 3} = \dfrac{6}{s^4(s - 3)}$.
b) $L\{te^{2t}\sin t\} = -\dfrac{d}{ds}L\{e^{2t}\sin t\} = -\dfrac{d}{ds}f(s - 2)$, where $f(s) = L\{\sin t\} = \dfrac{1}{s^2 + 1}$; $f(s - 2) = \dfrac{1}{(s - 2)^2 + 1}$. Therefore, $L\{te^{2t}\sin t\} = \dfrac{2(s - 2)}{((s - 2)^2 + 1)^2}$.

Q7. [20 pts.]

a) $L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$. The integral may be simplified by integration by parts. We obtain for s greater than some fixed s_0, $\int_0^\infty e^{-st} f'(t)\,dt = [e^{-st} f(t)]_0^\infty + s\int_0^\infty e^{-st}\, f(t)\,dt$, or
$L\{f'(t)\} = sL\{f(t)\} - f(0)$.
b) $L\{y\} = f(s)$, $L\{y'\} = sf(s) - y(0) = sf(s)$, $L\{y''\} = s^2 f(s) - sy(0) - y'(0) = s^2 f(s) - 1$. Therefore, our ODE gives: $s^2 f(s) - 1 + f(s) = \dfrac{s}{s^2+4}$, $\Longrightarrow$ $f(s)(s^2+1) = 1 + \dfrac{s}{s^2+4}$, $\Longrightarrow$
$f(s) = \dfrac{s}{(s^2+4)(s^2+1)} + \dfrac{1}{s^2+1}$, $\Longrightarrow$ $f(s) = \dfrac{As+B}{s^2+4} + \dfrac{Cs+D}{s^2+1} + \dfrac{1}{s^2+1}$ $\Longrightarrow$
$f(s) = -\dfrac{1}{3}\dfrac{s}{s^2+4} + \dfrac{1}{3}\dfrac{s}{s^2+1} + \dfrac{1}{s^2+1}$. Thus, $y(t) = L^{-1}\{f(s)\} = -\dfrac{1}{3}\cos 2t + \dfrac{1}{3}\cos t + \sin t$.

Q8. [10 pts.]

$H(t) = -1 - 2(t-3)u_3(t)$, so $L\{H(t)\} = -\dfrac{1}{s} - 2\dfrac{e^{-3s}}{s^2}$. Let $L\{y(t)\} = f(s)$. Then, $L\{y''(t)\} = s^2 f(s) - 1$. The ODE gives: $s^2 f(s) - 1 + f(s) = -\dfrac{1}{s} - 2\dfrac{e^{-3s}}{s^2}$, $\Longrightarrow$ $f(s) = \dfrac{1}{s^2+1} - \dfrac{1}{s(s^2+1)} - 2\dfrac{e^{-3s}}{s^2(s^2+1)}$ $\Longrightarrow$ $f(s) = \dfrac{1}{s^2+1} - \left(\dfrac{1}{s} - \dfrac{s}{s^2+1}\right) - 2e^{-3s}\left(\dfrac{1}{s^2} - \dfrac{1}{s^2+1}\right)$. Thus, $y(t) = L^{-1}\{f(s)\} = -1 + \cos t + \sin t - 2(t - 3 - \sin(t-3))u_3(t) = -1 + \cos t + \sin t + ((6-2t) + 2\sin(t-3))u_3(t)$. Or,

$$y(t) = \begin{cases} -1 + \cos t + \sin t, & 0 \le t \le 3 \\ 5 - 2t + \cos t + \sin t + 2\sin(t-3)) & t > 3. \end{cases}$$