Kuwait University
Faculty of Science
Department of Mathematics
Math 240: Differential equations
Final Exam, semester 2, 10 May 2018, 8–10am

Name: ______________________________

Student ID: ____________________

Section Number: ______ Serial Number: ______

Q1 (10 pts)	
Q2 (15 pts)	
Q3 (10 pts)	
Q4 (10 pts)	
Q5 (10 pts)	
Q6 (10 pts)	
Q7 (10 pts)	
Q8 (15 pts)	
Q9 (10 pts)	
Total	

1. (10pt) Solve the initial value problem

$$(y - x\sec x)dx + (\tan x)dy = 0; \quad y\left(\frac{\pi}{2}\right) = \frac{1}{8}.$$

Answer.

2. (10+ 5pt)

(a) Suppose $f(t)$ is continuous on $[0,1]$. Use variation of parameters to show that the general solution of $y'' + y = f(t)$ is

$$y = c_1 \cos t + c_2 \sin t + \int_0^t f(\beta)\sin(t-\beta)d\beta.$$

(b) By taking $f(t) = \dfrac{1}{\cos t}$ in part (a), find the general solution of $y'' + y = \dfrac{1}{\cos t}$.

Answer(a)

Answer(b)

3. (10pt) Find the general solution of $y'''(t) + 2y''(t) + y'(t) + 2y(t) = 5.$

Answer.

4. (8pt+ 2pt)

   (a) Obtain the first three nonzero terms in a power series solution about 0 for the initial value problem

$$3y''(x) + 2xy'(x) + y(x) = 0, \quad y(0) = 0,\ y'(0) = 1.$$

   (b) State the radius of convergence of the above series.

Answer(a)

Answer(b)

5. (10pt) Use the definition of the Laplace transform to prove that

$$\mathcal{L}\{\sinh 2t\} = \frac{2}{s^2 - 4}, \; s > 2.$$

Answer.

6. (5pt+5pt) Consider the initial value problem

$$y'' + 2y' = 10te^{3t}, \quad y(0) = 0, y'(0) = 0$$

(a) Find the Laplace transform $\mathcal{L}\{y(t)\}$.

(b) Find $y(t)$.

Answer(a)

Answer(b)

7. (5+5pt) Consider $y(t) = \begin{cases} 1+t^2, & 0 < t < 1 \\ 1+2t, & 1 \leqslant t < 2 \\ 5 & t \geqslant 2 \end{cases}$

(a) Express $y(t)$ in terms of unit step functions $u_c(t)$.

(b) Use part (a) to find $\mathcal{L}\{y(t)\}$.

Answer(a)

Answer(b)

8. (8pt+7pt)

   (a) Find $\mathcal{L}\{te^t \sin t \cos t\}$.

   (b) Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s(s^2+2s+5)}\right\}$.

Answer(a)

Answer(b)

9. (10pt) Solve the integral equation

$$y(t) = 4 + 2\int_t^0 \cos(t-\beta)y(\beta)d\beta.$$

Answer.

Solutions, final exam, math 240, May 2018

1. The equation is $y' + y\cot x = x\csc x$, which is linear with IF $\exp(\int \cot x dx) = \exp(\ln \sin x) = \sin x$. Thus $d(y\sin x) = x$, so $y\sin x = \frac{1}{2}x^2 + c$. Since $y = \frac{1}{8}$ when $x = \pi/2, c = (1-\pi^2)/8$.

2. (a) Since the roots of the characteristic equation are $\pm i$, try a particular solution of the form $y_p = a\cos t + b\sin t$. Then $a'\cos t + b'\sin t = 0, -a'\sin t + b'\cos t = f$, so that $b' = f\cos t, a' = -f\sin t$. Thus $y_p = -\cos t\int_0^t \sin\beta f(\beta)d\beta + \sin t\int_0^t \cos\beta f(\beta)d\beta = \int_0^t f(\beta)[\sin t\cos\beta - \cos t\sin\beta]d\beta = \int_0^t f(\beta)\sin(t-\beta)d\beta$. The general solution is $y = c_1\cos t + c_2\sin t + y_p$.

   (b) $y_p = \int_0^t \frac{1}{\cos\beta}\sin(t-\beta)d\beta = \int_0^t \frac{1}{\cos\beta}(\sin t\cos\beta - \cos t\sin\beta)d\beta$
   $= \sin t\int_0^t d\beta - \cos t\int_0^t \tan\beta d\beta = t\sin t + \cos t\ln\cos t$. The general solution is $y = c_1\cos t + c_2\sin t + y_p$.

3. The characteristic equation is $r^3 + 2r^2 + r + 2 = (r+2)(r^2+1) = 0$, so that $r = -2, \pm i$. For a particular solution take $y = a$. Then $2a = 5, a = 5/2$. Thus the general solution is $y = c_1e^{-2t} + c_2\cos t + c_3\sin t + \frac{5}{2}$.

4. (a) Put $y = \sum a_nx^n$. Substitution into the equation gives $3\sum n(n-1)a_nx^{n-2} + 2\sum na_nx^n + \sum a_nx^n = 0$. Thus $3(n+2)(n+1)a_{n+2} + 2na_n + a_n = 0$ for $n \geqslant 0$. This gives $6a_2 + a_0 = 0$ and $a_{n+2} = -\frac{2n+1}{3(n+2)(n+1)}a_n, n \geqslant 0$. We have $a_0 = y(0) = 0, a_1 = y'(0) = 1$. Thus $a_2 = 0 = a_4$ and $a_3 = -\frac{1}{6}, a_5 = \frac{7}{3\cdot 5\cdot 4}\frac{1}{6} = \frac{7}{360}$.
   Our solution is $y = x - x^3/6 + 7x^5/360 + \ldots$.

   (b) Since there are no singular points, the radius of convergence is infinite.

5. $L\{\sinh 2t\} = \int_0^\infty e^{-st}\frac{1}{2}(e^{2t} - e^{-2t})dt = \frac{1}{2}\int_0^\infty (e^{t(2-s)} - e^{t(-2-s)})dt = \frac{1}{2}\lim_{T\to\infty}\int_0^T (e^{t(2-s)} - e^{t(-2-s)})dt$, which for $s > 2$ is $\lim_{T\to\infty}\frac{1}{2}[e^{t(2-s)}/(2-s) + e^{t(-2-s)}/(2+s)]_0^T = \frac{1}{2}[-\frac{1}{2-s} - \frac{1}{2+s}] = \frac{2}{s^2-4}$, since $\lim_{T\to\infty} e^{T(2-s)} = 0 = \lim_{T\to\infty} e^{T(-2-s)}$.

6. (a) Put $L\{y\} = u$. Then $L\{y'\} = su, L\{y''\} = s^2u$ and $L\{e^{3t}\} = \frac{1}{s-3}$, so that $L\{te^{3t}\} = \frac{1}{(s-3)^2}$.

   The transform of our equation is $s^2u + 2su = \frac{10}{(s-3)^2}$, which gives
   $u = \frac{10}{s(s+2)(s-3)^2} = \frac{5}{9}\frac{1}{s} - \frac{1}{5}\frac{1}{s+2} - \frac{16}{45}\frac{1}{s-3} + \frac{2}{3}\frac{1}{(s-3)^2}$

   (b) By (a), $y = L^{-1}\{u\} = \frac{5}{9} - \frac{1}{5}e^{-2t} - \frac{16}{45}e^{3t} + \frac{2}{3}te^{3t}$.

7. (a) $y(t) = t^2 + 1 + u_1(t)(1 + 2t - (t^2+1)) + u_2(t)(5 - (1+2t))$
   $= t^2 + 1 - (t-1)^2u_1(t) + u_1(t) - 2(t-2)u_2(t)$.

(b) By (a), $L\{y(t)\} = \frac{2}{s^3} + \frac{1}{s} - 2\frac{e^{-s}}{s^3} + \frac{e^{-s}}{s} - 2\frac{e^{-2s}}{s^2}$.

8. (a) $L\{te^t \sin t \cos t\} = \frac{1}{2}L\{te^t \sin 2t\} = -\frac{1}{2}\frac{d}{ds}L\{(e^t \sin 2t)\} = -\frac{1}{2}\frac{d}{ds}F(s-1)$ where $F(s) = L\{\sin 2t\} = \frac{2}{s^2+4}$, $F(s-1) = \frac{2}{(s-1)^2+4}$, so that $\frac{d}{ds}F(s-1) = -\frac{4(s-1)}{((s-1)^2+4)^2}$ and $L\{te^t \sin t \cos t\} = \frac{2(s-1)}{((s-1)^2+4)^2}$.

(b) $\frac{1}{s(s^2+2s+5)} = \frac{a}{s} + \frac{bs+c}{s^2+2s+5} = \frac{1}{5}(\frac{1}{s} - \frac{s}{s^2+2s+5} - \frac{2}{s^2+2s+5}) = \frac{1}{5}(\frac{1}{s} - \frac{s+1}{(s+1)^2+4} - \frac{1}{(s+1)^2+4})$, consequently the inverse is $\frac{1}{5} - \frac{1}{5}e^{-t}\cos 2t - \frac{1}{10}e^{-t}\sin 2t$.

9. Put $L\{y(t)\} = Y(s)$. Then by the convolution theorem, $L\{\int_t^0 \cos(t-\beta)y(\beta)d\beta)\} = -L\{\int_0^t \cos(t-\beta)y(\beta)d\beta)\} = -\frac{sY}{s^2+1}$. The transform of our equation is $Y = \frac{4}{s} - 2\frac{sY}{s^2+1}$, so that $Y = 4\frac{s^2+1}{s(s+1)^2} = \frac{4}{s} - \frac{8}{(s+1)^2}$.
Thus $y(t) = L^{-1}\{Y(s)\} = 4 - 8te^{-t}$.