KUWAIT UNIVERSITY
Department of Mathematics

Math 240 | | 28/10/2014
Intro. ODE's | First In-Term Exam | Time: 90 Minutes

Electronic devices are not allowed during this exam.

1. (4 pts) Suppose $y(x)$ is a solution of the equation $y' = 2x^3\,(y+\sin y)$ with $y(0) = 0$. Use the existence and uniqueness of solutions of first order ODE's to show that $y(1) = 0$.

2. (8 pts) For each of the following differential equations, state whether it is linear or nonlinear and determine its order:

   (a) $y^{(4)} + \sqrt{x}\,y^{(3)} - \cos x = e^y$.

   (b) $3\,y' + (x+4)\,y = x^2 + y''$.

3. (8 pts) Consider the ODE $y' - \dfrac{1}{3}\,y = e^{-t}$.

   (a) Find the solution with $y(0) = a$.

   (b) Find the values of a for which (i) $y \to 0$ as $t \to \infty$, and (ii) $y \to -\infty$ as $t \to \infty$.

4. (16 pts) Given the differential equation:

$$(y\,e^{2xy} + x)\,dx + b\,x\,e^{2xy}\,dy = 0$$

   find the value of b for which the equation is exact and solve it for that value of b.

5. (48 pts) Solve the following differential equations:

   (a) $\big(2x + x\,y^2\,\sec^2(xy)\big)\,dx + \big(3y + x^2\,y\,\sec^2(xy)\big)\,dy = 0$.

   (b) $(x+1)\,dy - (y+1)\,dx = (x+1)\sqrt{y+1}\,dx$.

   (c) $\dfrac{dy}{dx} = -\dfrac{3x^2\,y + y^2}{2x^3 + 3x\,y}$ with $y(1) = 1$.

6. (16 pts) For the family of curves $x\,y = c$, where $c \neq 0$, find the member of its orthogonal trajectories that passes through the point $(0,1)$.

S O L U T I O N S

1. The problem $y' = f(x,y)$, $y(0) = 0$ has a unique solution since $f(x,y) = 2x^3\,(y + \sin y)$ and its y-derivative $f_y(x,y) = 2x^3\,(1 + \cos x)$ are continuous in every domain in $\mathbb{R}^2$ that contains the origin. In particular, the point $(1,0)$ is in one such domain and, by the *Existence and Uniqueness Theorem*, the problem $y' = f(x,y)$, $y(1) = 0$ also has a unique solution.

2. (a) nonlinear of order 4; (b) linear of second order.

3. $y' - y/3 = \mathrm{e}^{-t}$ is a linear DE. Therefore, taking the integrating factor $\rho = \mathrm{e}^{-t/3}$, we deduce the general solution

$$y(t) = \mathrm{e}^{t/3}\Big[\int \mathrm{e}^{-4t/3}\,dt + c\Big] = \mathrm{e}^{t/3}\Big[-\frac{3}{4}\,\mathrm{e}^{-4t/3} + c\Big]$$

The initial condition $y(0) = a$ gives $c = a + 3/4$ so that

$$y(t) = \big(a + \frac{3}{4}\big)\,\mathrm{e}^{t/3} - \frac{3}{4}\,\mathrm{e}^{-t}$$

$$\lim_{t\to\infty} y = 0 \quad\Rightarrow\quad a = -\frac{3}{4}\,; \qquad \lim_{t\to\infty} y = -\infty \quad\Rightarrow\quad a < -\frac{3}{4}$$

4. For the equation $(y\,\mathrm{e}^{2xy} + x)\,dx + b\,x\,\mathrm{e}^{2xy}\,dy = 0$ to be exact, we must have $b = 1$ since

$$M_y = \mathrm{e}^{2xy}\,(1 + 2xy) = b\,\mathrm{e}^{2xy}\,(1 + 2xy) = N_x \quad\Leftrightarrow\quad b = 1$$

Find a potential function $F(x,y)$:

$$F_x = y\,\mathrm{e}^{2xy} + x \quad\Rightarrow\quad F(x,y) = \frac{x^2}{2} + \frac{1}{2}\,\mathrm{e}^{2xy} + \varphi(y)$$

$$F_y = x\,\mathrm{e}^{2xy} \quad\Rightarrow\quad \varphi(y) = C_1$$

The general solution becomes $x^2 + \mathrm{e}^{2xy} = C$.

5. (a) By inspection: rewrite $\big(2x + x\,y^2\,\sec^2(xy)\big)\,dx + \big(3y + x^2\,y\,\sec^2(xy)\big)\,dy = 0$ as

$$2xdx + 3ydy + xy\sec^2(xy)\,(ydx + xdy) = 2xdx + 3ydy + (xy)\,\sec^2(xy)\,d(xy) = 0$$

Put $w = xy$ to reduce the above to

$$2xdx + 3ydy + w\,d\tan w = 0 \quad\Rightarrow$$

$$x^2 + \frac{3y^2}{2} + w\,\tan w + \ln|\cos w| = x^2 + \frac{3y^2}{2} + xy\,\tan(xy) + \ln|\cos(xy)| = C$$

(b) By setting $u = x + 1$ and $v = y + 1$, the equation becomes

$$u\,dv - v\,du = u\sqrt{v}\,du \quad\Rightarrow\quad \frac{dv}{du} - \frac{1}{u}\,v = \sqrt{v} \qquad \text{(Bernoulli)}$$

Put $w = v^{1-1/2} = \sqrt{v}$ to get

$$\frac{dw}{du} - \frac{1}{2u}\,w = \frac{1}{2} \quad \Rightarrow \quad w = u + C\sqrt{u}$$

hence, the general solution $\sqrt{y+1} = C\sqrt{x+1} + x + 1$.

(c) This equation is not exact. We have

$$M_y = 3x^2 + 2y\,; \quad N_x = 6x^2 + 3y\,; \quad \frac{M_y - N_x}{M} = -\frac{1}{y}$$

IF is $\rho(y) = \mathrm{e}^{\ln y} = y > 0$, and we obtain the exact equation:

$$(3x^2y^2 + y^3)\,dx + (2x^3y + 3xy^2)\,dy = 0$$

whose general solution is $x^3y^2 + xy^3 = C = 2$ from the initial condition $y(1) = 1$.

6. By implicit differentiation we get

$$\frac{dy}{dx} = -\frac{y}{x}$$

The OT's satisfy the DE

$$\frac{dy}{dx} = \frac{x}{y} \quad \Rightarrow \quad y^2 = x^2 + K = x^2 + 1$$

under the condition $y(0) = 1$.