Kuwait University	Math 240	20/12/2014
Dept. of Mathematics	Final Exam	Duration: 2 Hours

Calculators and communication devices are not allowed.

1. (5 pts. each) Answer with True or False and justify your answers:

   (a) $y^2\,e^{x-1} + 3x^2 + (2y\,e^{x-1} + x)\,y' = 0$ is an exact equation.

   (b) All solutions of $\dfrac{dy}{dx} = 1 + y^2$ are valid for all x.

   (c) If y_1 and y_2 are linearly independent solutions of $y'' + \cos x\, y = 0$, then their Wronskian W satisfies $\dfrac{dW}{dx} = 0$ and $W = c$, where c is a nonzero constant.

2. (5 pts. each) Given an equation of the form $\dfrac{dy}{dx} = \dfrac{1}{y}\, f\left(\dfrac{y^2}{x}\right)$:

   (a) Show that by setting $u = \dfrac{y^2}{x}$, this equation can be transformed into a separable equation.

   (b) Use part (a) to solve the equation $\dfrac{dy}{dx} = \dfrac{y^3}{x^2} - \dfrac{y}{2x}$.

3. (10 pts.) Solve the initial value problem $y\,dx = (x + \sqrt{y^2 - x^2})\,dy\,,\ y(0) = 1\,.$

4. (10 pts.) Find the Laplace transform $f(s) = \mathcal{L}\,[F(t)]$ of the function

$$F(t) = \begin{cases} t\,, & 0 \le t < \pi/2 \\ \cos t\,, & t \ge \pi/2 \end{cases}$$

5. (10 pts.) Find the inverse Laplace transform $F(t) = \mathcal{L}^{-1}[f(s)]$ of the function

$$f(s) = \frac{s-2}{s^2 - 4s + 5}\, e^{-3s}$$

6. (15 pts.) Use the Laplace transform to solve the initial value problem

$$y''(t) + y(t) = \sin t\,, \qquad y(0) = y'(0) = 0\,.$$

7. (15 pts.) Find the Laplace transform $f(s) = \mathcal{L}\,[y(t)]$, where

$$y'(t) - y(t) = t \int_0^t (t - \beta)\, e^{-\beta}\, d\beta\,, \qquad y(0) = 1\,.$$

8. (15 pts.) Determine the first five nonzero terms in the power series solution about $x = 0$ for the initial value problem

$$(4 - x^2)\,y'' + 2y = 0\,,\ y(0) = y'(0) = 1\,.$$

   What is the interval of validity (convergence) of this power series solution?

S O L U T I O N S

1. (a) False: $M_y = 2y\,e^{x-1} \neq 2y\,e^{x-1} + 1 = N_x$.
(b) False: $y' = 1 + y^2 \;\to\; \tan^{-1} y = x + c \;\to\; y = \tan(x+c)$, not valid when $x + c = \pm\,(2n+1)\,\pi/2$.
(c) True: $W = y_1\,y_2' - y_1'\,y_2 \;\to\; W' = y_1\,y_2'' - y_1''\,y_2 = -y_1\,y_2\,\cos x + y_1\,y_2\,\cos x = 0$, so that $W = c \neq 0$ since y_1, y_2 are linearly independent.

2. (a) $u = y^2/x \;\to\; x\,u = y^2 \;\to\; x\,u' + u = 2y\,y' = 2\,f(u)$, a separable equation.
(b)

$$\frac{dy}{dx} = \frac{y^3}{x^2} - \frac{y}{2x} = \frac{1}{y}\left(\frac{y^4}{x^2} - \frac{y^2}{2x}\right) \;\to\; f(u) = u^2 - \frac{1}{2}\,u$$

$$x\,u' + u = 2u^2 - u \;\to\; \frac{du}{u^2 - u} = 2\,\frac{dx}{x} \;\to\; \frac{u-1}{u} = c\,x^2 \;\to\; y^2 = \frac{x}{1 - c\,x^2}$$

3. Put $x = yz \;\to\; dx = y\,dz + z\,dy$

$$\frac{dx}{dy} = \frac{x + \sqrt{y^2 - x^2}}{y} = \frac{x}{y} + \sqrt{1 - \frac{x^2}{y^2}} \;\to\; y\,\frac{dz}{dy} + z = z + \sqrt{1 - z^2} \;\to$$

$$\frac{dz}{\sqrt{1 - z^2}} = \frac{dy}{y} \;\to\; \sin^{-1}\frac{x}{y} = \ln y + c$$

where $y(0) = 1$ gives $c = 0$.

4. $F(t) = t + \alpha(t - \pi/2)\,\big[\cos t - t\big] = t - \alpha(t - \pi/2)\,\big[\sin(t - \pi/2) + (t - \pi/2) + \pi/2\big]$

$$f(s) = \frac{1}{s^2} - e^{-\pi s/2}\left[\frac{1}{s^2+1} + \frac{1}{s^2} + \frac{\pi}{2\,s}\right]$$

5.

$$\mathcal{L}^{-1}\left[e^{-3s}\,\frac{(s-2)}{(s-2)^2+1}\right] = \mathcal{L}^{-1}\,\left[e^{-3s}\,\mathcal{L}\left[e^{2t}\,\cos t\right]\right] = \alpha(t-3)\,e^{2(t-3)}\,\cos(t-3)$$

6. Take LT of the equation:

$$s^2\,f(s) + f(s) = \frac{1}{s^2+1} \;\to\; f(s) = \frac{1}{(s^2+1)^2} = \mathcal{L}\left[\sin * \sin(t)\right]$$

$$y(t) = \mathcal{L}^{-1}[f(s)] = \int_0^t \sin(t-\beta)\,\sin\beta\,d\beta = \frac{1}{2}\int_0^t \big[\cos(t - 2\beta) - \cos t\big]\,d\beta = \cdots$$

and the rest is calculus: $y(t) = (\sin t - t\,\cos t)/2$.

7. Take LT of the equation:

$$(s-1)\,\mathcal{L}\left[y(t)\right] - 1 = \mathcal{L}\left[t\int_0^t (t-\beta)\,e^{-\beta}\,d\beta\right] = -\frac{d}{ds}\,\mathcal{L}\left[t\right]\mathcal{L}\left[e^{-t}\right] = -\frac{d}{ds}\,\frac{1}{s^2\,(s+1)}$$

$$(s-1)\,\mathcal{L}\left[y(t)\right] = 1 - \frac{d}{ds}\left[\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right] = 1 + \frac{2}{s^3} - \frac{1}{s^2} + \frac{1}{(s+1)^2}$$

$$f(s) = \mathcal{L}\left[y(t)\right] = \frac{1}{s-1}\left[1 + \frac{2}{s^3} - \frac{1}{s^2} + \frac{1}{(s+1)^2}\right] = \frac{s^5 + 2s^4 + s^3 + 3s + 2}{s^3(s-1)(s+1)^2}$$

8.

$$y=\sum_{n=0} a_n\, x^n \;\rightarrow\; y'=\sum_{n=1} n\, a_n\, x^{n-1} \;\rightarrow\; y''=\sum_{n=2} n(n-1)\, a_n\, x^{n-2}$$

$$(4-x^2)\, y''+2y=0 \;\rightarrow$$

$$(8a_2+2a_0)+(24a_3+2a_1)\, x+\sum_{n=2}\big[4(n+2)(n+1)\, a_{n+2}-(n+1)(n-2)\, a_n\big]\, x^n=0$$

Since $a_0=a_1=1$, we obtain $a_2=-\frac{1}{4}$, $a_3=-\frac{1}{12}$ and

$$a_{n+2}=\frac{n-2}{4(n+2)}\, a_n\,, \quad n\geq 2$$

so that $a_4=a_6=a_8=\cdots=0$, whereas the terms with odd indices are nonzero: $a_5=-1/240$, etc.

$$y(x)=1+x-\frac{1}{4}\, x^2-\frac{1}{12}\, x^3-\frac{1}{240}\, x^5-\cdots$$

The solution is valid for $|x|<2$.