Kuwait University	**Math 240**	**Date:**	**22 April 2014**
Department of Mathematics	**Second Exam**	**Time:**	**6:30-8pm**

Calculators and mobile phones are not allowed.
Answer all questions.

1. (5pts each) Let

$$y_1 = x^3 \text{ and } y_2 = |x|^3 \qquad -\infty < x < \infty$$

   (a) Compute the Wronskian of $\{y_1, y_2\}$ on $(-\infty, \infty)$.

   (b) Determine whether the set $\{y_1, y_2\}$ is linearly dependent or independent.

   (c) Can y_1 be a solution of an equation $y'' + p(x)y' + q(x)y = 0$ on $(-\infty, \infty)$, where p and q are continuous on $(-\infty, \infty)$? Explain your answer.

2. (20pts + 5pts)

   (a) Use the exponential shift to solve the initial value problem $(2D-1)^3 y = e^x, y(0) = 0, y'(0) = 0, y''(0) = 0$.

   (b) Find a homogeneous linear differential equation satisfied by $y = 3x^2 + 4e^{-x}\sin 5x$.

3. (20pts)

   (a) Solve the initial value problem $y'' - 4y = 4 + 5\sin x,\ y(0) = -1, y'(0) = c$, where c is a real number.

   (b) Find c so that the solution in part (a) is bounded on $(-\infty, \infty)$.

4. (20pts) Find the general solution of the equation $(x^2 - 1)y'' - 2xy' + 2y = 0, x \neq \pm 1$, given that $y_1 = x$ is a solution.

5. (20pts) Solve the differential equation $y'' + y = 1 + \cot x; \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$.

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1. (a) Both functions are twice differentiable on $\mathbb{R}$ and $W(y_1, y_2) = \begin{vmatrix} x^3 & -x^3 \\ 3x^2 & -3x^2 \end{vmatrix} = 0$ for $x < 0$ and $W(y_1, y_2) = \begin{vmatrix} x^3 & x^3 \\ 3x^2 & 3x^2 \end{vmatrix} = 0$ for $x \geq 0$

   (b) Suppose $ay_1 + by_2 = 0$ for all x. Substitute $x = \pm 1$ to obtain $a \pm b = 0$. Thus $a = b = 0$ and $\{y_1, y_2\}$ is linearly independent.

   (c) Since $y_1(0) = 0 = y_1'(0)$, by the Existence and Uniqueness Theorem for homogeneous linear ODE, if y_1 were a solution of the given ODE, we would have $y_1 = 0$, the zero function, contradiction.

2. (a)

$$\begin{aligned} (2D-1)^3 y &= e^x \Rightarrow e^{-\frac{x}{2}} \left(D - \frac{1}{2}\right)^3 y = \frac{1}{8} e^{\frac{x}{2}} \Rightarrow D^3 e^{-\frac{x}{2}} y = \frac{1}{8} e^{\frac{x}{2}} \\ e^{-\frac{x}{2}} y &= e^{\frac{x}{2}} + c_1 x^2 + c_2 x + c_3 \\ y &= e^x + \left(c_1 x^2 + c_2 x + c_3\right) e^{\frac{x}{2}} \end{aligned}$$

   Applying the conditions $y(0) = 0 = y'(0) = y''(0)$, gives
   $c_3 = -1, c_2 = -1/2, c_1 = -1/8$.

   (b) The roots of the characteristic equation will be $0, 0, 0$, from x^2 and $-1 \pm 5i$ from $e^{-x} \sin 5x$. Thus equation is $D^3((D+1)^2 + 25)y = 0$.

3. (a) The characteristic equation has roots ± 2, so $y_c = ae^{-2x} + be^{2x}$.
   Try $y_p = A + B \sin x$. Substitution into the ODE gives $y_p = -1 - \sin x$. Thus the general solution is
   $y = ae^{-2x} + be^{2x} - 1 - \sin x$. Since $y(0) = -1, y'(0) = c$, we have
   $a + b = 0, -2a + 2b - 1 = c$, so that $a = -\frac{1}{4}(1+c), b = \frac{1}{4}(1+c)$.

   (b) Since y is to be bounded on $\mathbb{R}$, we must have $a = b = 0$, so $c = -1$ and $y = -1 - \sin x$.

4. Put $y = xu$. Then $y' = u + xu', y'' = 2u' + xu''$. Substitution into the ODE gives $(x^3 - x)u'' = 2u'$. Let $w = u'$, so that $\frac{dw}{w} = \frac{2}{x(x-1)(x+1)} = -\frac{2}{x} + \frac{1}{x-1} + \frac{1}{x+1}$. Thus $\ln w = \ln \frac{a(x^2-1)}{x^2}$, so that $w = u' = a(1 - \frac{1}{x^2})$ and $u = a(x + \frac{1}{x}) + b$. Finally, $y = a(1 + x^2) + bx, a, b \in \mathbb{R}$.

5. Variation of parameters

   $y_p = A \cos x + B \sin x = 0$
   $A' \cos x + B' \sin x = 0 \quad \& \quad -A' \sin x + B' \cos x = 1 + \cot x$. Thus
   $\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} 0 \\ 1 + \cot x \end{pmatrix}$,
   $\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix} \begin{pmatrix} 0 \\ 1 + \cot x \end{pmatrix}$
   $= \begin{pmatrix} -(1 + \cot x) \sin x \\ (1 + \cot x) \cos x \end{pmatrix} = \begin{pmatrix} -\sin x - \cos x \\ \cos x + \csc x - \sin x \end{pmatrix}$.
   $A = \cos x - \sin x, \qquad B = \sin x + \int \frac{\cos^2 x}{\sin x} dx = \sin x + \cos x + \ln|\cot x - \csc x|$
   $y_p = (\cos x - \sin x) \cos x + (\sin x + \cos x + \ln|\cot x - \csc x|) \sin x = 1 + (\sin x) \ln|\cot x - \csc x|$
   $y = c_1 \cos x + c_2 \sin x + (\sin x) \ln|\cot x - \csc x| + 1$.