Kuwait University	Math 240	Date:	10th March 2014
Department of Mathematics	First Exam	Time:	6:30-8pm

Calculators and mobile phones are not allowed.
Answer all questions.

1. (4pts) State which of the following differential equations is linear or non-linear. Determine the order of the ODE.

   (a) $y'' + y' - \sin(t+y) = \sin t$.

   (b) $(y''')^2 + ty' + y\cos t = \ln t$.

2. (4pts) Determine whether all solutions of the ODE

$$dy + (1+y^4)dx = 0$$

   are increasing or decreasing functions. Justify your answer.

3. (8pts) Determine whether $y = e^{x^2} + e^{x^2}\int_0^x e^{-t^2}dt$ is a solution of the equation $y' - 2xy = 1$.

4. (16pts each) Solve the following differential equations:

   (a) $y' = \dfrac{y\sin x - \sin y}{x\cos y + \cos x}$.

   (b) $(1+\cos x)y' - \sin x - e^{-y}\sin x = 0$ with $y(0) = 0$.

   (c) $\left(x^5 - y^2\right)dx + 2xydy = 0$.

   (d) $(x+y-2)\,dx - (x-4y-2)\,dy = 0$.

5. (20pts) For the family $y - x + 1 = ce^{-x}$, find the member of the orthogonal trajectories which passes through the point $(0,0)$.

Solutions of Math 240 exam 1, held on 10th March 2014

1. (a) Not linear, order 2.
   (b) Not linear, order 3.
2. $y' = -1 - y^4 < 0$ for all x, so decreasing.
3. By the product rule, chain rule and the FTC, we have
   $y' = e^{x^2}(e^{-x^2}) + 2xe^{x^2}(1 + \int) = 1 + 2xy$, so y is a solution.
4. (a) The equation can be written as
   $(y \sin x - \sin y)dx - (x \cos y + \cos x)dy = 0$, so
   $M_y = \sin x - \cos y = N_x$. Our equation is exact and the potential function, f, satisfies $f_x = y \sin x - \sin y$, so
   $f = -y \cos x - x \sin y + g(y)$ and
   $f_y = -\cos x - x \cos y + g' = -x \cos y - \cos x$, so the solution is $y \cos x + x \sin y = C$.
   (b) The equation is $(1 + \cos x)dy = (1 + e^{-y}) \sin x dx$, which is separable and becomes $\frac{e^y dy}{1+e^y} = \frac{\sin x dx}{1+\cos x}$. Integration gives $(1 + e^y)(1 + \cos x) = c$. Since $y(0) = 0, c = 4$ and $(1 + e^y)(1 + \cos x) = 4$.
   (c) The equation transforms to $y' - \frac{1}{2x}y = -\frac{1}{2}x^4 y^{-1}$, a Bernoulli equation. Putting $u = y^2$ gives the linear equation $u' - \frac{u}{x} = -x^4$ with IF $1/x$ and solution $u/x = -x^4/4 + c$ or $y^2 = -x^5/4 + cx$.
   (d) The lines $x + y = 2, x - 4y = 2$ intersect at $(2, 0)$. Putting $u = x - 2$ gives $(u + y)du - (u - 4y)dy = 0$, which is homogeneous. If we take $u = yv$, we have the separable equation $(1 + v)ydv + (v^2 + 4)dy = 0$, which integrates to $\frac{1}{2}\tan^{-1}(v/2) + \frac{1}{2}\ln(v^2 + 4) + \ln y = C$ where $v = \frac{x-2}{y}$.
5. Differentiating the equation gives $y' - 1 = -ce^{-x} = x - y - 1$, so the orthogonal trajectories are given by $-\frac{1}{y'} - 1 = x - y - 1$, or $y'(y - x) = 1$. Putting $u = y - x$ gives $u(1 + u') = 1$, which on separation of variables is $(\frac{1}{1-u} - 1)du = dx$. Integration gives $-u - \ln|1 - u| = x + k$ and applying the condition $y(0) = 0$ gives $x - y + 1 = e^{-y}$.