Kuwait University	**Math 240**	**Date: 10 May 2014**
Department of Mathematics	**Final Exam**	**Time: 5-7pm**

Calculators and mobile phones are not allowed.
Answer all questions.

1. (10pt) Solve $(x^3y^2 + 4y)dx + (x^2y^3 + 4x)dy = 0$, where $y(1) = 1$.

2. (10pt) Find the general solution of the differential equation $y'' - 2y' + y = \dfrac{e^x}{1+x^2}$.

3. (10pt) Solve the IVP $y''' + y'' + y' + y = e^x, y(0) = 0, y'(0) = 0, y''(0) = 0$.

4. (14pt) Find the Laplace transform of the following functions.

   (a) $\cosh t \sin^2 t$

   (b) $t \displaystyle\int_0^t \beta^5 e^{2(t-\beta)} d\beta$

5. (21pt) Find the inverse Laplace transform of the following functions.

   (a) $\dfrac{3s^2 + 5s - 1}{s\,(s^2 + 2s + 2)}$

   (b) $\dfrac{s^2 - 4s + 3}{(s^2 - 4s + 5)^2}$

   (c) $\tan^{-1}\left(\dfrac{1}{s}\right)$ *(Hint: Use $L\{tF(t)\} = -f'(s)$.)*

6. (15pt) Solve the initial value problem $y'' + y = F(t), \quad y(0) = 0 = y'(0)$, where

$$F(t) = \begin{cases} 1, & 0 < t < 1 \\ -2t + 1, & t \geq 1. \end{cases}$$

7. (20pt) Consider a power series $y = \sum_{n=0}^{\infty} a_n x^n$ solution of the equation $(x^2 - 2)y'' + 6xy' + 6y = 0$.

   (a) Show that $a_{n+2} = \dfrac{n+3}{2(n+1)} a_n, n \geq 0$.

   (b) Find the power series solution satisfying $y(0) = 1, y'(0) = 0$ and state where the solution is valid.

1. The equation can be written as $(xy)^2(xdx+ydy)+4d(xy)=0$, so that $xdx+ydy+4d(xy)/(xy)^2=0$ which integrates to $\frac{1}{2}(x^2+y^2)-4/(xy)=c$. Since $y=1$ when $x=1$, we have $c=-3$.

2.

$$\begin{aligned}
y''-2y'+y &= 0 \Rightarrow y_c = c_1e^x + c_2xe^x \\
\text{V. of parameters} &: \quad y_p = Ae^x + Bxe^x \\
\begin{pmatrix} A' \\ B' \end{pmatrix} &= \begin{pmatrix} e^x & xe^x \\ e^x & (1+x)e^x \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \frac{e^x}{1+x^2} \end{pmatrix} = e^{-x}\begin{pmatrix} 1+x & -x \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ \frac{e^x}{1+x^2} \end{pmatrix} = \begin{pmatrix} -\frac{x}{x^2+1} \\ \frac{1}{x^2+1} \end{pmatrix} \\
\begin{pmatrix} A \\ B \end{pmatrix} &= \begin{pmatrix} -\frac{1}{2}\ln\left(1+x^2\right) \\ \tan^{-1}x \end{pmatrix} \Rightarrow y_p = \left(-\frac{1}{2}\ln\left(1+x^2\right)+x\tan^{-1}x\right)e^x \\
y &= y_c + y_p = c_1e^x + c_2xe^x + \left(x\tan^{-1}x - \frac{1}{2}\ln\left(1+x^2\right)\right)e^x
\end{aligned}$$

3. Characteristic equation is $(m+1)(m^2+1)=0$, so $y_c = Ae^{-x}+B\sin x+C\cos x$ and by inspection $y_p=\frac{1}{4}e^x$. Thus $y=Ae^{-x}+B\sin x+C\cos x+\frac{1}{4}e^x$. The initial conditions give $A+C+\frac{1}{4}=0, -A+B+\frac{1}{4}=0, A-C+\frac{1}{4}=0$. This gives $A=-\frac{1}{4}, B=-\frac{1}{2}, C=0$.

4. (a)

$$\begin{aligned}
\cosh t\sin^2 t &= \frac{1}{4}\left(e^t+e^{-t}\right)(1-\cos 2t) \\
L\left\{\cosh t\sin^2 t\right\} &= \frac{1}{4}\left[L\{1-\cos 2t\}\right]_{s\to s-1} + \frac{1}{4}\left[L\{1-\cos 2t\}\right]_{s\to s+1} \\
&= \frac{1}{4}\left(\frac{1}{s-1}-\frac{s-1}{(s-1)^2+4}\right)+\frac{1}{4}\left(\frac{1}{s+1}-\frac{s+1}{(s+1)^2+4}\right) \\
&= \frac{1}{(s-1)\left((s-1)^2+4\right)}+\frac{1}{(s+1)\left((s+1)^2+4\right)} \\
&= \frac{1}{(s-1)\left(s^2-2s+5\right)}+\frac{1}{(s+1)\left(s^2+2s+5\right)}
\end{aligned}$$

(b)

$$\begin{aligned}
L\left\{t\int_0^t \beta^5 e^{2(t-\beta)}d\beta\right\} &= -\frac{d}{ds}L\left\{\int_0^t \beta^5 e^{2(t-\beta)}d\beta\right\} = -\frac{d}{ds}L\left\{t^5\right\}L\left\{e^{2t}\right\} \\
&= -\frac{d}{ds}\frac{5!}{s^6}\frac{1}{s-2} = -5!\left[\frac{-1}{s^7}\frac{1}{s-2}-\frac{1}{s^6}\frac{1}{(s-2)^2}\right] = \frac{120\,(7s-12)}{s^7\,(s-2)^2}
\end{aligned}$$

5. (a)

$$\begin{aligned}
\frac{3s^2+5s-1}{s\left(s^2+2s+2\right)} &= \frac{Bs+C}{s^2+2s+2}+\frac{A}{s} = \frac{7s+12}{2\left(s^2+2s+2\right)}-\frac{1}{2s} = \frac{7s+12}{2\left((s+1)^2+1\right)}-\frac{1}{2s} \\
L^{-1}\left\{\frac{3s^2+5s-1}{s\left(s^2+2s+2\right)}\right\} &= \frac{1}{2}e^{-t}L^{-1}\left\{\frac{7(s-1)+12}{s^2+1}\right\}-\frac{1}{2} = \frac{1}{2}e^{-t}L^{-1}\left\{\frac{7s+5}{s^2+1}\right\}-\frac{1}{2} \\
&= \frac{7}{2}e^{-t}\cos t+\frac{5}{2}e^{-t}\sin t-\frac{1}{2}
\end{aligned}$$

(b)

$$\begin{aligned}
& L^{-1}\left\{\frac{s^2-4s+3}{(s^2-4s+5)^2}\right\} \\
=\ & L^{-1}\left\{\frac{1}{(s-2)^2+1}\right\}-L^{-1}\left\{\frac{2}{((s-2)^2+1)^2}\right\} \\
=\ & e^{2t}L^{-1}\left\{\frac{1}{s^2+1}\right\}-e^{2t}L^{-1}\left\{\frac{2}{(s^2+1)^2}\right\} = e^{2t}\sin t - e^{2t}L^{-1}\left\{\frac{2}{(s^2+1)^2}\right\} \\
=\ & e^{2t}\sin t - 2e^{2t}\int_0^t \sin\beta\sin(t-\beta)d\beta = e^{2t}\sin t - e^{2t}\int_0^t\left[\cos(t-\beta-\beta)-\cos(\beta+t-\beta)\right]d\beta \\
=\ & e^{2t}\sin t - e^{2t}\int_0^t\left[\cos(t-2\beta)-\cos t\right]d\beta = e^{2t}\sin t - e^{2t}\left[\frac{1}{2}\sin t+\frac{1}{2}\sin t - t\cos t\right] \\
=\ & te^{2t}\cos t
\end{aligned}$$

Alternatively

$$\begin{aligned} L^{-1}\left\{\frac{s^2-4s+3}{(s^2-4s+5)^2}\right\} &= L^{-1}\left\{\frac{(s-2)^2-1}{((s-2)^2+1)^2}\right\} = e^{2t}L^{-1}\left\{\frac{s^2-1}{(s^2+1)^2}\right\} = -e^{2t}L^{-1}\left\{\frac{d}{ds}\frac{s}{s^2+1}\right\} \\ &= te^{2t}L^{-1}\frac{s}{s^2+1} = te^{2t}\cos t \end{aligned}$$

(c)

$$\begin{aligned} L^{-1}\left\{\tan^{-1}\frac{1}{s}\right\} &= F(t) \Rightarrow \tan^{-1}\frac{1}{s} = L\{F(t)\} \\ L\{tF(t)\} &= -\left(L\{F(t)\}\right)' = -\left(\tan^{-1}\frac{1}{s}\right)' = \frac{1}{s^2\left(1+\frac{1}{s^2}\right)} = \frac{1}{s^2+1} \\ tF(t) &= L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t \Rightarrow L^{-1}\left\{\tan^{-1}\frac{1}{s}\right\} = \frac{\sin t}{t} \end{aligned}$$

6. Use the unit step function to write the DE as $y''+y = 1-2t\alpha(t-1)$. Put $Y = Y(s) = L\{y\}$ and apply the Laplace transform to both sides of the DE to get

$$s^2Y - sy(0) - y'(0) + Y = \frac{1}{s} - 2e^{-s}L\{t+1\} = \frac{1}{s} - 2e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right)$$

$$\begin{aligned} Y &= \frac{1}{s(s^2+1)} - 2e^{-s}\frac{1+s}{s^2(s^2+1)} \\ Y &= \frac{1}{s} - \frac{s}{s^2+1} - 2e^{-s}\frac{1+s}{s^2(s^2+1)} && \text{Partial Fractions: } \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1} \\ y &= 1-\cos t - 2\alpha(t-1)L^{-1}\left\{\frac{1+s}{s^2(s^2+1)}\right\}_{t\to t-1} && \text{Partial Fractions: } \frac{1+s}{s^2(s^2+1)} = \frac{1}{s}+\frac{1}{s^2}-\frac{s+1}{s^2+1} \\ &= 1-\cos t - 2\alpha(t-1)L^{-1}\left\{\frac{1}{s}+\frac{1}{s^2}-\frac{1}{s^2+1}-\frac{s}{s^2+1}\right\}_{t\to t-1} \\ &= 1-\cos t - 2\alpha(t-1)\{1+t-1-\sin(t-1)-\cos(t-1)\} \\ &= 1-\cos t - 2\alpha(t-1)\{t-\sin(t-1)-\cos(t-1)\} \end{aligned}$$

7. (a) Substituting into the equation gives
$(x^2-2)\sum n(n-1)a_nx^{n-2} + 6x\sum na_nx^{n-1} + 6\sum a_nx^n = 0 \implies$
$\sum[-2(n+2)(n+1)a_{n+2} + (n^2-n+6n+6)a_n]x^n = 0$
$\implies 2(n+2)(n+1)a_{n+2} = a_n(n+2)(n+3)$
$\implies a_{n+2} = \frac{n+3}{2(n+1)}a_n$.

(b) The initial conditions entail $a_1 = y'(0) = 0$ and inductively $a_{2m+1} = 0$ for all $m \geq 0$. Also $a_0 = y(0) = 1, a_2 = \frac{3}{2}, a_{2m} = (\frac{1}{2})^m(2m+1)$, so $y = \sum_{m=0}(\frac{1}{2})^m(2m+1)x^{2m}$.
The series converges up to the nearest singularity, that is, for $|x| < \sqrt{2}$.