Kuwait University	**Math 240**	**Date:**	**4th December 2013**
Department of Mathematics	**2nd Midterm Exam**	**Time:**	**6.30-8pm**

Calculators and mobile phones are not allowed
Each question or part carries 20 marks.

1. Show that if f, f' and f'' are continuous functions on $[a, b]$ and $f(x) \neq 0$ for all $x \in [a, b]$, then f, xf and $x^2 f$ are linearly independent on $[a, b]$.

2. Use the exponential shift property, $e^{ax} f(D)y = f(D - a)(e^{ax} y)$, to find the general solution of the equation
$(D^3 - 3D^2 + 3D - 1)y = 2e^x \sec^2 x \tan x$.

3. Use the method of undetermined coefficients to solve the initial value problem
$(D^2 - 1)y = e^{-x}(2\sin x + 4\cos x),\ y(0) = y'(0) = 0$.

4. (a) Verify that $y = e^x$ is a solution of the equation
$xy'' - (2x + 1)y' + (x + 1)y = 0, x > 0$.
Use this fact and the method of reduction of order to find the general solution of
$xy'' - (2x + 1)y' + (x + 1)y = e^x$.

   (b) Use variation of parameters to solve the equation
$xy'' - (2x + 1)y' + (x + 1)y = \frac{2e^x}{x^2},\ x > 0$,
given that $y = x^2 e^x$ is a solution of the associated homogeneous differential equation.

Solutions, 240, exam 2, December 2013

1. $W[f, xf, x^2f] = \begin{vmatrix} f & xf & x^2f \\ f' & f + xf' & 2xf + x^2f' \\ f'' & 2f' + xf'' & 2f + 4xf' + x^2f'' \end{vmatrix} = 2f^3 \neq 0, x \in [a, b].$

2. $(D^3 - 3D^2 + 3D - 1)y = (D-1)^3y$ and $e^{-x}(D-1)^3y = D^3(e^{-x}y)$, so our equation is,
$D^3(e^{-x}y) = 2\sec^2 x \tan x, \implies D^2(e^{-x}y) = 2\int \sec^2 x \tan x dx = C_1 + \tan^2 x, \implies$
$D(e^{-x}y) = C_1x + \int \tan^2 x dx = C_2 + C_1x + \tan x - x, \implies$
$e^{-x}y = C_3 + C_2x + C_1x^2/2 + \int(\tan x - x)dx = C_3 + C_2x + C_4x^2 - \ln|\cos x|, \implies$
$y = e^x(C_3 + C_2x + C_4x^2 - \ln|\cos x|).$

3. The equation $(D^2 - 1)y = 0$ has solution $y_c = c_1e^{-x} + c_2e^x$ and we note that
$(D^2 + 2D + 2)[e^{-x}(2\sin x + 4\cos x)] = 0.$
The equation
$(D^2 - 1)(D^2 + 2D + 2)y = 0$, has solution $y = c_1e^{-x} + c_2e^x + c_3e^{-x}\cos x + c_4e^{-x}\sin x$
and we can take
$y_p = e^{-x}(A\cos x + B\sin x)$. Substitution gives
$y_p'' - y_p = e^{-x}(2\sin x + 4\cos x) \iff 2A - B = 2, A + 2B = -4 \implies A = 0, B = -2,$
$y = c_1e^{-x} + c_2e^x - 2e^{-x}\sin x,$
$y(0) = 0 \implies c_1 + c_2 = 0$, and $y'(0) = 0 \implies c_1 - c_2 = -2$, so, $c_2 = 1, c_1 = -1$ and
finally, $y = e^x - e^{-x} - 2e^{-x}\sin x$.

4. (a) $y = ve^x$, so $y' = e^xv + e^xv', y'' = e^xv + 2e^xv' + e^xv''$, and substitution shows $xy'' - (2x+1)y' + (x+1)y = 0$.
The non-homogeneous equation becomes $xv'' - v' = 1$.
Put $w = v'$. Then $w' - \frac{1}{x}w = \frac{1}{x}$, so $w = x[c + \int x^{-2}dx] = cx - 1$. Now,
$v' = cx - 1 \implies v = cx^2/2 + c_1 - x$ and $y = ve^x$.

(b) By part (a), e^x is a solution of the homogeneous equation so we can suppose
$y_p = Ae^x + Bx^2e^x$. Substitution gives
$A'e^x + B'x^2e^x = 0$ and
$A'e^x + B'(2xe^x + x^2e^x) = \frac{2e^x}{x^3}$.
Thus $B' = 1/x^4 \implies B = -\frac{1}{3x^3}$ and $A' = -x^2B'$ which leads to $A' = -1/x^2$, so
$A = 1/x$.
Thus $y_p = \frac{2e^x}{3x}$ and finally, $y = c_1e^x + c_2x^2e^x + \frac{2e^x}{3x}$.

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1. Let $[a, b]$ be an interval and x_0 a fixed number in $[a, b]$. Suppose y is a solution of the equation
$y'' + P(x)y' + Q(x)y = 0,$
subject to the constraints $y(x_0) = y'(x_0) = 0$. Use the Existence and Uniqueness Theorem to prove that $y(x) = 0$ for every $x \in [a, b]$.

2. Use the exponential shift property, $e^{ax}f(D)y = f(D-a)(e^{ax}y)$, to solve $(D^2 + 4D + 4)y = e^{-2x}\sin x$.

3. Use the method of undetermined coefficients to solve the equation
$y'' + 4y' + 3y = 15e^{2x} + e^{-x}.$

4. Verify that $y = e^x$ is a solution of the equation
$(x-1)y'' - xy' + y = 0, x > 1.$
Use this fact and the method of reduction of order to find the general solution of
$(x-1)y'' - xy' + y = 1,\ x > 1.$

5. Use variation of parameters to solve the equation
$(1-x)y'' + xy' - y = 2(x-1)^2e^{-x},\ x > 1,$
given that $y = x$ is a solution of the associated homogeneous differential equation.