KUWAIT UNIVERSITY
Department of Mathematics

First Midterm Exam — 30 October 2013
Math 240 — Time: 90 minutes

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*Calculators and mobile phones are not allowed*

1. (10 pts) Show that $\varphi(x) = \dfrac{1}{x}$ is a solution of the nonlinear equation

$$y' = \frac{1}{x^2} - \frac{y}{x} - y^2$$

Use the substitution $y = \dfrac{1}{x} + \dfrac{1}{u(x)}$ to transform the above equation into a linear equation in $u(x)$ of order one, then solve for $u(x)$.

2. (30 pts) Find the general solutions of the the following equations:

   (a) $y\,dx + x\,(\ln y - \ln x - 2)\,dy = 0$

   (b) $\dfrac{y'}{y} + \dfrac{xy+1}{2y-x} = 0$

   (c) $(ay - by^2)\,dx - dy = 0$, where $a$ and $b$ are positive constants.

3. (20 pts) Determine the particular solution of the initial value problem

$$(2x + 3y + 1)\,dx + (4x + 6y + 1)\,dy = 0\,, \qquad y(-2) = 2$$

4. (20 pts) Given a family of curves $y\,(x^2+1) = c\,x$, where $c$ is an arbitrary constant. Find the orthogonal trajectory of this family which passes through the point $(\sqrt{2}, \sqrt{2})$.

5. (20 pts) Find the equation of the curve $C$ for which the $y$-intercept of the tangent line at each point $(x, y) \neq (0, 0)$ on it is equal to $2x\,y^2$.

S O L U T I O N S

1. First part is easy. $y' = \varphi' - u'/u^2$ and the DE becomes

$$u' - \frac{3}{x}\,u = 1 \quad \to \quad u = -\frac{x}{2} + C\,x^3$$

2. Find the general solutions of the the following equations:

(a) Divide by $x$ and put $y = v\,x$

$$v\,dx + (\ln v - 2)\,(v\,dx + x\,dv) = 0 \quad \to \quad v\,(\ln v - 1)\,dx + x\,(\ln v - 2)\,dv = 0 \quad \to$$

$$\int \frac{dx}{x} + \int \frac{\ln v - 2}{\ln v - 1}\,\frac{dv}{v} = c_1 \quad \to \quad \int \frac{dx}{x} + \int \frac{z-1}{z}\,dz = c_1\,; \quad (z = \ln v - 1)$$

$$\ln x + z - \ln z = c_1 \quad \to \quad \ln x + \ln v - 1 - \ln(\ln v - 1) = c_1 \quad \to$$

$$\ln y - \ln(\ln\frac{y}{x} - 1) = 1 + c_1 \quad \to \quad y = C\,(\ln\frac{y}{x} - 1)$$

(b) The equation is rewritten as $(xy^2 + y)\,dx + (2y - x)\,dy = 0$, which is not exact. But $M_y - N_x = 2xy + 2$, so that $M^{-1}\,(M_y - N_x) = 2y^{-1}$. The integrating factor becomes $\rho = y^{-2}$ and the general solution is found to be

$$x^2 + 2\,\frac{x}{y} + \ln y^4 = C$$

(c) The equation becomes $y' + ay = -by^2$ (Bernoulli). Take $u = y^{-1}$ to get $u' - au = b$ whose solution is

$$u = \frac{1}{y} = \mathrm{e}^{ax}\left[c - \frac{b}{a}\,\mathrm{e}^{-ax}\right] \quad \to \quad y = \frac{a}{a\,c\,\mathrm{e}^{ax} - b}$$

3. The two lines making up the coefficients are parallel. Put $z = 2x + 3y$ so that $dz = 2dx + 3dy$ and

$$(z+1)\,(dz - 3dy) + 2(2z+1)\,dy = 0$$

which leads to the separable equation

$$\frac{z+1}{z-1}\,dz = -dy \quad \to \quad z + 2\,\ln|z-1| = -y + c \quad \to \quad 2x + 3y + 2\,\ln|2x+3y-1| = c - y$$

The initial condition gives $c = 4$.

4. Differentiate the equation to get $y'(x^2+1) + 2xy = c$. Then,

$$y' = \frac{c - 2xy}{x^2+1}\,; \qquad c = \frac{y}{x}\,(x^2+1)$$

which simplifies to

$$y' = \frac{y\,(1-x^2)}{x\,(1+x^2)}$$

For OT's we have

$$y' = -\frac{x\,(1+x^2)}{y\,(1-x^2)} \quad \to \quad y\,dy = \frac{x\,(x^2+1)}{x^2-1}\,dx = \left[x + \frac{2x}{x^2-1}\right] dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + \ln|x^2-1| + C$$

The particular OT passing through $(\sqrt{2}, \sqrt{2})$ will satisfy $1 = 1 + \ln 1 + C$, so that $C = 0$ and this OT is given by $y^2 = x^2 + 2\ln|x^2-1|$.

5. The equation of the tangent line at any point $(x, y)$ on $C$ is $y = y'\,x + b$, where $b$ is the $y$-intercept which depends on the point $(x, y)$. This dependency is given by $b = 2x\,y^2$. Thus, we have to solve the equation

$$y = y'\,x + 2x\,y^2 \quad \to \quad y' - \frac{1}{x}\,y = -2y^2$$

which is a Bernoulli equation. Put $u = y^{-1}$ to get

$$u' + \frac{1}{x}\,u = 2 \quad \to \quad u = \frac{x^2+c}{x} \quad \to \quad y = \frac{x}{x^2+c}$$