Kuwait University	**Math 240**	**Date: 21st December 2013**
Department of Mathematics	**Final Exam**	**Time: 5-7pm**

Calculators and mobile phones are not allowed

1. (10pts each) Find the general solution of the differential equations:

   (a) $(1 + x^2)y' + 6xy = x$,

   (b) $y'' - 2y' + y = \frac{e^t}{t}$, $t > 0$.

2. (10pts) Suppose $y_1(t)$ is a solution of $y' + p(t)y = 0$ and $y_2(t)$ is a solution of $y' + p(t)y = g(t)$. Show that $y_1 + y_2$ is also a solution of $y' + p(t)y = g(t)$.

3. (10pts) Determine the family G of orthogonal trajectories of the family F of curves $x^2 + xy + y^2 = C$.

4. (15pts) Given that $-\pi/2 < y < \pi/2$, find the particular solution of the problem
   $(x - \sin y)dy + \tan y\, dx = 0, y(1) = \pi/6$.

5. (7pts + 8pts) Let $\alpha(t)$ be the step function $\alpha(t) = \begin{cases} 0 \text{ if } t < 0, \\ 1 \text{ if } t \geq 0. \end{cases}$

   and $f(s) = L\{F(t)\}$ the Laplace transform of $F(t)$.

   (a) Show that for any real number $c \geq 0$,
   $L\{\alpha(t - c)F(t - c)\} = e^{-cs}f(s), t \geq c$.

   (b) Use the result in (a) to solve the initial value problem:
   $$y''(t) + y(t) = g(t) = \begin{cases} 0 & \text{if } t < \frac{\pi}{2}, \\ \cos t & \text{if } t \geq \frac{\pi}{2} \end{cases},$$
   $$y(0) = 1, y'(0) = 2$$

6. (15pts) Determine the solution of the equation
   $y''(t) = \int_0^t (t - \tau)y(\tau)d\tau$ that satisfies $y(0) = 0, y'(0) = 1$.

7. (15pts) Determine the first five non-zero terms of the power series solution of the differential equation
   $(4 + x^2)y'' + xy' - 9y = 0$ that satisfies $y(0) = \frac{8}{9}, y'(0) = 3$.

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1. (a) $(1+x^2)y'+6xy=x \iff y'+\frac{6x}{1+x^2}y=\frac{x}{1+x^2}$. An IF is $\exp\int\frac{6x}{1+x^2}dx=(1+x^2)^3$, so $d[y(1+x^2)^3]=x(1+x^2)^2dx$ and $y=\frac{1}{6}+c/(1+x^2)^3$.

   (b) The general solution of the corresponding homogeneous equation is $y_c=c_1e^t+c_2te^t$ and we search for a particular solution of the form $y_p=Ae^t+Bte^t$, where $A'e^t+B'te^t=0, A'e^t+B'(e^t+te^t)=e^t/t$. This gives $B'=1/t$, so $B=\ln t$ and $A'=-1$, so $A=-t$. Thus the general solution is $y=c_1e^t+c_2te^t-te^t+te^t\ln t$.

2. $[y_1+y_2]'+p[y_1+y_2]=y_1'+py_1+y_2'+py_2=g$.

3. $2x+y+xy'+2yy'=0$ gives $y_F'=-\frac{2x+y}{x+2y}$, so $y_G'=\frac{x+2y}{2x+y}$ which becomes $(x+2y)dx-(2x+y)dy=0$: this is homogeneous. Setting $y=xv$ gives $(x+2xv)dx-(2x+xv)(xdv+vdx)=0$, which leads to $(1-v^2)dx-x(2+v)dv=0$, so $\frac{dx}{x}=\frac{2+v}{1-v^2}dv$ or $2\frac{dx}{x}=(\frac{3}{1-v}+\frac{1}{1+v})dv$. This integrates to $x^2=c\frac{|1+v|}{|1-v|^3}$ and $|x-y|^3=c|x+y|$.

4. Here $M=\tan y, N=x-\sin y$ and $\frac{1}{M}(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x})=\tan y$, so the equation is not exact, but there is an IF $\mu=\exp(-\int\tan y dy)=\cos y>0$. Thus we get the exact equation $\cos y(x-\sin y)dy+\sin y dx=0$ which becomes $d(x\sin y)-\frac{1}{2}\sin 2y dy=0$. This integrates to $x\sin y+\frac{1}{4}\cos 2y=c$. The initial conditions lead to $c=5/8$ and $x\sin y+\frac{1}{4}\cos 2y=\frac{5}{8}$.

5. (a) $L[\alpha(t-c)F(t-c)]=\int_0^\infty e^{-st}\alpha(t-c)F(t-c)dt=\int_c^\infty e^{-st}F(t-c)dt$ $=\int_0^\infty e^{-s(c+u)}F(u)du=e^{-sc}\int_0^\infty e^{-su}F(u)du=e^{-cs}f(s)$.

   (b) Application of the Laplace transform and initial conditions leads to $(1+s^2)f(s)-s-2=-L[\alpha(t-\frac{\pi}{2})\sin(t-\frac{\pi}{2})]$ since $\cos t=-\sin(t-\frac{\pi}{2})$. From (a), $(1+s^2)f(s)-s-2=-\frac{e^{-\pi s/2}}{1+s^2}$. By the convolution Theorem and part (a), we obtain $y=2\sin t+\cos t-\int_0^{t-\frac{\pi}{2}}\sin(t-\tau)\sin\tau d\tau$.

6. By the Laplace transform and initial conditions, we find
$L[y(t)]=\frac{s^2}{s^4-1}$. Using partial fractions we obtain
$L[y]=\frac{1}{2}\frac{1}{1+s^2}+\frac{1}{4}\frac{1}{s-1}-\frac{1}{4}\frac{1}{s+1}$.
Hence $y=\frac{1}{2}\sin t+\frac{1}{4}(e^t-e^{-t})$.

7. We search for a solution of the form $y=\sum_{n=0}^\infty a_nx^n$ where $a_0=8/9$ and $a_1=3$. We have $y'=\sum_{n=1}^\infty na_nx^{n-1}, y''=\sum_{n=2}^\infty n(n-1)a_nx^{n-2}$.
Substituting into the equation gives
$-9a_0+8a_2+(24a_3-8a_1)x+\sum_{n=2}^\infty[(n^2-9)a_n+4(n+1)(n+2)a_{n+2}]x^n=0$.
Thus $a_2=\frac{9}{8}a_0=1, a_3=\frac{1}{3}a_1=1$ and for $n\geq 2$,
$a_{n+2}=\frac{9-n^2}{4(1+n)(2+n)}a_n$.
In particular, $a_4=\frac{5}{48}$ and $y=\frac{8}{9}+3x+x^2+x^3+\frac{5}{48}x^4+\cdots$