Kuwait University	Math. 240	June 24, 2013
Dept. of Mathematics	Second Midterm Exam	Duration 90 Minutes

Calculators of any kind are not allowed.

All communication devices must be switched off.

Answer the following questions.

1. [5 Pts] (a) Show that the functions $u_1 = t\sin\left(\frac{1}{t}\right)$ and $u_2 = t\cos\left(\frac{1}{t}\right)$ are linearly independent on the interval $(0, \infty)$.

   [5 Pts] (b) Find $A = (xD - 1)(D + 2x)$.

2. [15 Pts] Use the exponential shift rule to find the general solution of the equation

$$y'' - 4y' + 4y = e^{2x}\sec^2 x$$

3. [15 Pts] Solve the initial value problem

$$y''' - y'' + 4y' - 4y = 0;\ y(0) = -4,\ y'(0) = 1 \text{ and } y''(0) = 1.$$

4. Consider the nonhomogeneous linear equation $F(D)y = R(x)$, where

$$F(D) = \left(D^2 + D - 2\right)\left(D^3 + D + 10\right)$$

   and

$$R(x) = 5 - 3xe^{-2x} + 2xe^{3x}\cos 3x$$

   (a) [10 Pts] Find the general solution of the homogeneous linear equation $F(D)y = 0$.

   (b) [10 Pts] For the nonhomogeneous equation $F(D)y = R(x)$, **set up** the appropriate form of a particular solution y_p, but **do not determine** the values of the coefficients.

5. [20 Pts] Find the general solution of the differential equation

$$x^2y'' - xy' + y = 6x\ln x;\ x > 0$$

   given that $y_1 = x$ is a solution of the corresponding homogeneous equation.

6. [20 Pts] Use the method of **variation of parameters** to find the general solution of the differential equation

$$y'' + y = \cot x$$

1.a $W(u_1,u_2) = \begin{vmatrix} t\sin\left(\frac{1}{t}\right) & t\cos\left(\frac{1}{t}\right) \\ \sin\left(\frac{1}{t}\right) - \frac{1}{t}\cos\left(\frac{1}{t}\right) & \cos\left(\frac{1}{t}\right) + \frac{1}{t}\sin\left(\frac{1}{t}\right) \end{vmatrix} = 1 \neq 0$

1.b

$$\begin{aligned} Ay &= (xD-1)(D+2x)y = (xD-1)(y'+2xy) \\ &= xD(y'+2xy) - (y'+2xy) = x(y''+2y+2xy') - y' - 2xy \\ &= xy'' + \left(2x^2-1\right)y' = \left[xD^2 + \left(2x^2-1\right)D\right]y \end{aligned}$$

$$\boxed{A = xD^2 + \left(2x^2-1\right)D}$$

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2. $\left(D^2-4D+4\right)y = e^{2x}\sec^2 x \implies e^{-2x}(D-2)^2 y = \sec^2 x \implies D^2\left(e^{-2x}y\right) = \sec^2 x$

$D\left(e^{-2x}y\right) = \tan x + c_1 \implies e^{-2x}y = \ln|\sec x| + c_1x + c_2$

$$\boxed{y = [c_1x + c_2 + \ln|\sec x|]\,e^{2x}.}$$

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3. $m^3 - m^2 + 4m - 4 = 0 \implies (m-1)\left(m^2+4\right) = 0 \implies m = 1,\ \pm 2i$

$y = c_1e^x + c_2\cos 2x + c_3\sin 2x$

$y' = c_1e^x - 2\sin 2x + 2c_3\cos 2x$ and $y'' = c_1e^x - 4c_2\cos 2x - 4c_3\sin 2x$

$y(0) = -4,\ y'(0) = 1$ and $y''(0) = 1 \implies c_1 + c_2 = -4,\ c_1 + 2c_3 = 1$ and

$c_1 - 4c_2 = 1$ respectively. Solving this linear system we obtain $c_1 = -3,\ c_2 = -1$ and $c_3 = 2$

$$\boxed{y = -3e^x - \cos 2x + 2\sin 2x.}$$

4.a $\left(m^2+m-2\right)\left(m^3+m+10\right) = 0 \implies (m-1)(m+2)(m+2)\left[\left((m-1)^2+4\right)\right] = 0$

$m = 1,\ -2,\ -2,\ 1 \pm 2i$

$$\boxed{y_c = c_1e^x + c_2e^{-2x} + c_3xe^{-2x} + c_4e^x\cos 2x + c_5e^x\sin 2x.}$$

4.b $m' = 0,\ -2,\ -2,\ 3 \pm 3i,\ 3 \pm 3i$

$$\boxed{y_p = A + \left(Bx^2 + Cx^3\right)e^{-2x} + (E + Fx)\,e^{3x}\cos 3x + (G + Hx)\,e^{3x}\sin 3x.}$$

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5. $y = y_1 u = xu, \qquad y' = xu' + u, \qquad y'' = xu'' + 2u'$

$$u'' + \left(\frac{1}{x}\right) u' = \frac{6\ln x}{x^2} \implies \mu = \exp\left(\int \frac{dx}{x}\right) = x$$

$$xu' = 6\int x\left(\frac{\ln x}{x^2}\right) dx + c_1 = 6\int \left(\frac{\ln x}{x}\right) dx + c_1 = 3(\ln x)^2 + c_1$$

$$u = 3\int \frac{(\ln x)^2}{x} dx + c_1 \int \frac{dx}{x} + c_2 = (\ln x)^3 + c_1 \ln x + c_2$$

$$\boxed{y = x\left[c_1 \ln x + c_2 + (\ln x)^3\right].}$$

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6. $y = y_c + y_p$

$m^2 + 1 = 0 \implies \boxed{y_c = c_1 \cos x + c_2 \sin x.}$

$y_p = A\cos x + B\sin x$

$$\begin{cases} A'\cos x + B'\sin x &= 0 \\ A'(-\sin x) + B'\cos x &= \cot x \end{cases} \implies \begin{cases} A' &= -\cos x \\ B' &= \dfrac{\cos^2 x}{\sin x} = \csc x - \sin x \end{cases}$$

$$\begin{cases} A &= -\sin x \\ B &= \ln|\csc x - \cot x| + \cos x \end{cases}$$

$$\boxed{y_p = \sin x \ln|\csc x - \cot x|.}$$
