Kuwait University	Math. 240	July 16, 2013
Department of Mathematics	Final Examination	Duration 2 hours

Calculators of any kind are not allowed. All communication devices must be switched off.

Answer the following questions.

1. [15 Pts] Find the particular solution of the equation

$$\left(4+2y-3\sin^2 x\right)dx+2\tan x\,dy=0;\ y\left(\frac{\pi}{2}\right)=1$$

2. [15 Pts] Use the method of undetermined coefficients to find the general solution of the equation

$$y''+4y'+5y=3e^{-2x}+4\cos x$$

3. [10 Pts] Find the general solution of the differential equation

$$x^2y''-3xy'+3y=x^4e^x;\ x>0$$

given that $y_1=x$ is a solution of the corresponding homogeneous equation.

4. Evaluate

(a) [7 Pts] $L\left\{\int_0^t \frac{e^\beta}{\sqrt{\beta}}d\beta\right\}$.

(b) [8 Pts] $L^{-1}\left\{\frac{(3s-2)\,e^{-2s}}{s^2-2s+2}\right\}$.

5. (a) [10 Pts] For $a>0$, show that from $L^{-1}\{f(s)\}=F(t)$ it follows that

$$L^{-1}\{f(as+b)\}=\frac{1}{a}\exp\left(-\frac{bt}{a}\right)F\left(\frac{t}{a}\right)$$

(b) [5 Pts] Use part (a) to find $L^{-1}\left\{\frac{1}{(2s-3)^3}\right\}$.

6. [15 Pts] Use Laplace transform to solve the following equation with the condition $y(0)=1$

$$y'(t)+\int_0^t y(\beta)\,d\beta=F(t);\ \text{where } F(t)=\begin{cases}0, & 0\le t<1\\ t, & t\ge 1\end{cases}$$

7. [15 Pts] Find the particular solution near the origin of the equation

$$\left(1+x^2\right)y''-4xy'+6y=0;\ y(0)=1 \text{ and } y'(0)=-3$$

See backside for formulas

TABLE FOR LAPLACE TRANSFORMS

$F(t)$	$f(s)$	$F(t)$	$f(s)$
1	$\frac{1}{s}$	$\sin bt$	$\frac{b}{s^2+b^2}$
e^{at}	$\frac{1}{s-a}$	$\cos bt$	$\frac{s}{s^2+b^2}$
t^n	$\frac{n!}{s^{n+1}}$	$\sinh bt$	$\frac{b}{s^2-b^2}$
t^x	$\frac{\Gamma(x+1)}{s^{x+1}}$	$\cosh bt$	$\frac{s}{s^2-b^2}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$		

FORMULAS FOR LAPLACE TRANSFORMS

$$L\{F(t)\} = f(s)$$

$$L\{t^n F(t)\} = (-1)^n f^{(n)}(s)$$

$$L\{F'(t)\} = sf(s) - F(0)$$

$$L\{e^{at} F(t)\} = f(s-a)$$

$$L\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)$$

$$L\{F(t-c)\,\alpha(t-c)\} = e^{-cs} f(s)$$

$$L\{F'''(t)\} = s^3 f(s) - s^2 F(0) - sF'(0) - F'(0)$$

$$L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)\,du$$

$$L\left\{\int_0^t F(u)\,G(t-u)\,du\right\} = f(s)\,g(s)$$

$$L\left\{\int_0^t F(u)\,du\right\} = \frac{f(s)}{s}$$

1. $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right)=\dfrac{2-2\sec^2 x}{2\tan x}=-\tan x \implies \mu(x)=\exp\left(-\int \tan x dx\right)=\cos x$

$\left(4\cos x+2y\cos x-3\sin^2 x\cos x\right)dx+2\sin x dy=0$ (Exact Eq.) $\implies \left(4+2y-\sin^2 x\right)\sin x=c$

$y\left(\frac{\pi}{2}\right)=1 \implies c=5 \implies \boxed{\left(4+2y-\sin^2 x\right)\sin x=5.}$

OR $y'+(\cot x)y=\left(-2+\frac{3}{2}\sin^2 x\right)\cot x$: (Linear Eq) $\implies \mu(x)=\exp\left(\int \cot dx\right)=\sin x$

$(\sin x)y=\int\left(-2+\frac{3}{2}\sin^2 x\right)\cot x\sin x dx+c=-2\sin x+\frac{1}{2}\sin^3 x+c$

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2. $y=y_c+y_p$

$m^2+4m+5=0 \implies (m+2)^2+1=0 \implies m=-2\pm i$

$\boxed{y_c=e^{-2x}\left(c_1\cos x+c_2\sin x\right)}$

$y_p=Ae^{-2x}+B\sin x+C\cos x \implies$

$y'=-2Ae^{-2x}-B\sin x+C\cos x$ and $y''=4Ae^{-2x}-B\cos x-C\sin x$

$Ae^{-2x}+4(B+C)\cos x+4(-B+C)\sin x=3e^{-2x}+4\cos x \implies A=3,\ B=C=\frac{1}{2}.$

$\boxed{y_p=3e^{-2x}+\frac{1}{2}\sin x+\frac{1}{2}\cos x}$

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3. $y=y_1v=xv, \qquad y'=xv'+v, \qquad y''=xv''+2v'$

$v''-\left(\dfrac{1}{x}\right)v'=xe^x \implies \mu=\exp\left(\int -\dfrac{dx}{x}\right)=\dfrac{1}{x}$

$\dfrac{1}{x}v'=\int\dfrac{1}{x}\left(xe^x\right)dx+c_1=e^x+c_1$

$v=\int xe^x dx+c_1\int x dx+c_2=(x-1)e^x+kx^2+c_2;\ k=\frac{1}{2}c_1.$

$\boxed{y=x\left[kx^2+c_2+(x-1)e^x\right].}$

4.a $L\left\{\int_0^t \dfrac{e^\beta}{\sqrt{\beta}}d\beta\right\}=\dfrac{f(s)}{s};\ f(s)=L\left\{\dfrac{e^t}{\sqrt{t}}\right\}=\dfrac{\sqrt{\pi}}{\sqrt{s-1}} \implies \boxed{L\left\{\int_0^t \dfrac{e^\beta}{\sqrt{\beta}}d\beta\right\}=\dfrac{\sqrt{\pi}}{s\sqrt{s-1}}.}$

4.b $L^{-1}\left\{\dfrac{(3s-2)e^{-2s}}{s^2-2s+2}\right\}=L^{-1}\left\{\dfrac{[3(s-1)+1]e^{-2(s-1)}e^{-2}}{(s-1)^2+1}\right\}$

$=e^{t-2}L^{-1}\left\{\dfrac{(3s+1)e^{-2s}}{s^2+1}\right\}=\boxed{e^{t-2}\left[3\cos(t-2)+\sin(t-2)\right]\alpha(t-2).}$

5.a $f(s) = L\{F(t)\} = \int\limits_0^\infty e^{-st} F(t)\, dt \Longrightarrow$

$$L\left\{\frac{1}{a}\exp\left(-\frac{bt}{a}\right) F\left(\frac{t}{a}\right)\right\} = \int\limits_0^\infty e^{-st}\left[\frac{1}{a}e^{-bt/a}F\left(\frac{t}{a}\right)\right] dt$$

$$= \int\limits_0^\infty e^{-(as+b)t/a}\left[F\left(\frac{t}{a}\right)\right] d\left(\frac{t}{a}\right) = \int\limits_0^\infty e^{-(as+b)u}F(u)\, du;\ u = \frac{t}{a}$$

$$= f(as+b)$$

5.b $a = 2,\ b = -3,\ f(s) = \dfrac{1}{s^3} \implies F(t) = \frac{1}{2}t^2$

$$L^{-1}\{f(as+b)\} = \frac{1}{a}\exp\left(-\frac{bt}{a}\right) F\left(\frac{t}{a}\right)$$

$$L^{-1}\left\{\frac{1}{(2s-3)^3}\right\} = \frac{1}{2}\exp\left(\frac{3t}{2}\right)\left(\frac{1}{2}\right)\left(\frac{t}{2}\right)^2 = \boxed{\frac{1}{16}t^2\exp\left(\frac{3t}{2}\right).}$$

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6. $F(t) = t\alpha(t-1) = [(t-1)+1]\,\alpha(t-1) \implies L\{F(t)\} = \left(\dfrac{1}{s^2}+\dfrac{1}{s}\right)e^{-s}$

$L\{y(t)\} = u(s) \implies L\{y'(t)\} = su - 1$

$$su - 1 + \frac{u}{s} = \left(\frac{1}{s^2}+\frac{1}{s}\right)e^{-s} \implies u(s) = \frac{s}{s^2+1} + \left(\frac{1}{s} - \frac{s}{s^2+1} + \frac{1}{s^2+1}\right)e^{-s}$$

$$\boxed{y(t) = \cot t + [1 - \cos(t-1) + \sin(t-1)]\,\alpha(t-1).}$$

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7. $y = \sum\limits_{n=0} a_n x^n \implies y' = \sum\limits_{n=1} n a_n x^{n-1}$ and $y'' = \sum\limits_{n=2}(n-1) n a_n x^{n-2}$.

$y(0) = 1 \implies a_0 = 1,\ y'(0) = -3 \implies a_1 = -3$

$$\sum_{n=2}(n-1)na_nx^{n-2} + \sum_{n=2}(n-1)na_nx^n - 4\sum_{n=1}na_nx^n + 6\sum_{n=0}a_nx^n = 0$$

$$\sum_{n=0}(n+1)(n+2)a_{n+2}x^n + \sum_{n=1}(n-2)(n-3)a_nx^n = 0$$

$$a_{n+2} = -\frac{(n-2)(n-3)}{(n+1)(n+2)};\ n = 0,\ 1,\ 2,\ \ldots$$

$$\boxed{y = a_0\left(1-3x^2\right) + a_1\left(x - \tfrac{1}{3}x^3\right) = 1 - 3x - 3x^2 + x^3.}$$
