Kuwait University	Math 240	11th March 2013, 6.30–8pm
Department of Mathematics	First Midterm Exam	20 Marks

Calculators and mobile phones are not allowed

1. (15pts each.) Find the general solutions of the equations:-

   (a) $x\dfrac{dy}{dx} = y(1 + \ln y - \ln x),\ \ x > 0, y > 0.$

   (b) $\dfrac{dy}{dx} = y - x - 1 + (x - y + 2)^{-1}.$

   (c) $(1 + ye^x + xye^x)dx + (xe^x + 2)dy = 0.$

   (d) $(4x + 3y^2)dx + 2xy\,dy = 0.$

2. (20pts.) Determine the particular solution of the initial value problem:-

$$y' + y\ln y = \frac{y}{(\ln y)^2},\ y(0) = 1.$$

3. (20pts.) Find the value of λ such that the parabolas $y = C_1x^2 + \lambda$ are orthogonal trajectories of the family of ellipses $x^2 + 2y^2 - y = C_2$, where C_1 and C_2 are parameters.

Solutions to 240, Exam 1, March 2013

1. (a) Since $x \neq 0$, we have $\frac{dy}{dx} = \frac{y}{x}(1 + \ln \frac{y}{x})$ and the equation is homogeneous, so put $z = y/x$. Then $y' = z + xz'$ and our equation becomes

   $z + xz' = z(1 + \ln z)$ or $xz' = z \ln z$. This gives $\frac{dz}{z \ln z} = \frac{dx}{x}$ which integrates to $\ln \ln z = \ln x + \ln c$, so $\ln z = cx$ and $y = xe^{cx}$.

   (b) Put $z = y - x$. Then $y' = 1 + z'$ and our equation becomes $1 + z' = z - 1 + \frac{1}{2-z}$ which reduces to $z' = \frac{(z-3)(z-1)}{z-2}$, so that $\frac{z-2}{(z-1)(z-3)}dz = dx$ or $\frac{1}{2}[\frac{1}{z-1} + \frac{1}{z-3}]dz = dx$, which integrates to $\frac{1}{2}(\ln|z-1| + \ln|z-3|) = x + c$ or $|(z-1)(z-3)| = ce^{2x}$. Finally $|(y-x-1)(y-x-3)| = ce^{2x}$.

   (c) Here $M_y = e^x + xe^x = N_x$, so our equation is exact and there exists F with $F_x = 1 + ye^x + xye^x$. Thus

   $F = x + ye^x + y \int xe^x dx + h(y) = x + xye^x + h.$

   Next, $F_y = xe^x + h'$, so $h' = 2, h = 2y$ and our solution is

   $x + 2y + xye^x = c.$

   (d) We have $\frac{M_y - N_x}{N} = \frac{6y-2y}{2xy} = 2/x$, so an IF is $\exp(\int \frac{2}{x}dx) = x^2$. Applying the IF gives $(4x^3 + 3x^2y^2)dx + 2yx^3dy = 0$ which we re-arrange as

   $4x^3dx + (3x^2y^2dx + 2yx^3dy) = 0$, so $4x^3dx + d(x^3y^2) = 0$ which integrates to $x^4 + x^3y^2 = c$.

2. Put $z = \ln y$. Then our ODE becomes $z' + z = z^{-2}$, a Bernoulli equation. Putting $u = z^3$ gives $\frac{u'}{3z^2} + z = \frac{1}{z^2}$, which leads to $u' + 3u = 3$. This is linear with an IF e^{3x}. Applying the IF gives $e^{3x}u' + 3ue^{3x} = 3e^{3x}$ which integrates to $ue^{3x} = e^{3x} + c$. When $x = 0, y = 1, z = 0, u = 0$, so $c = -1$ and our solution is $e^{3x}(\ln y)^3 = e^{3x} - 1$.

3. For the parabolas, $y' = 2c_1x = \frac{2(y-\lambda)}{x}$.

   We can write the ellipse family as $x^2 + 2(y - \frac{1}{4})^2 = c_2 + \frac{1}{8}$ which leads to $y' = \frac{x}{\frac{1}{2} - 2y}$.

   For the families to be orthogonal, the product of the slopes must be -1, so

   $\frac{2(y-\lambda)}{x}\frac{x}{\frac{1}{2}-2y} = -1$ and $\lambda = \frac{1}{4}$.