Kuwait University, Department of Mathematics

MATH 240, DIFFERENTIAL EQUATIONS

11 May, 2013, 5-7pm. Final Exam.

Answer all questions. Calculators and mobile phones are NOT allowed.

1. (10pts) Solve the equation
   $(y\sin x - y^3)dx - 2\cos x dy = 0, 0 < x < \frac{\pi}{2}$.

2. (10pts) Find the particular solution of the equation
   $y(y+3x+2)dx + x(y+x+1)dy = 0$ such that $y(1) = 2$.

3. (10pts) Show that $y = e^x$ is a solution of the equation
   $xy'' - 2(1+x)y' + (2+x)y = 0$
   and find the general solution of the non-homogeneous equation
   $xy'' - 2(1+x)y' + (2+x)y = (x-2)e^x$.

4. (10pts)

   (a) If $f(s) = L[F(t)]$ is the Laplace transform of $F(t)$, show that
   $f(\lambda s) = L\left[\frac{1}{\lambda}F\left(\frac{t}{\lambda}\right)\right], \lambda > 0;$

   (b) Consider the periodic function $F(t), t \geq 0$, of period $\omega = 2$ such that
   $F(t) = \begin{cases} 3, & 0 \leq t \leq 1 \\ 0, & 1 < t \leq 2 \end{cases}$
   Find $L[F(t)]$.

5. (15pts)

   (a) If $F(t) = L^{-1}\{f(s)\}$ is the inverse Laplace transform of $f(s)$, show that $L^{-1}[\frac{f(s)}{s}] = \int_0^t F(y)dy$.

   (b) By using part(a), find $L^{-1}\left[\frac{n!}{s(s-a)^{n+1}}\right]$, where $a \in \mathbb{R}$ and $n \geq 0$ is an integer.

6. (15pts) Use the Laplace transform to solve the initial value problem
   $y''(t) + y(t) = F(t), y(0) = y'(0) = 0$
   where
   $F(t) = \begin{cases} \cos t, & 0 \leq t \leq \pi \\ 0, & t > \pi \end{cases}$

7. (15pts) Solve the integral equation $y(t) = t + \int_0^t y(\beta)\sin 2(t-\beta)d\beta$.

8. (15pts) Find the first four non-zero terms of the power series solution of the initial value problem
   $(1+x)y'' - y = 0, y(0) = 0, y'(0) = 1$.

Solutions, Final 240 exam, May 2013

1. $2\cos x dy = (y\sin x - y^3)dx \implies y' = \frac{1}{2}\tan x y - \frac{1}{2\cos x}y^3$, a Bernoulli equation. Putting $u = 1/y^2$ gives $u' + u\tan x = 1/\cos x \implies$ IF $= e^{\int \tan x dx}$ and $u = \sin x + C\cos x \implies y = 1/\sqrt{\sin x + C\cos x}$.

2. $M = y(y+3x+2), N = x(y+x+1)$ and $\frac{1}{N}(\partial M/\partial y - \partial N/\partial x) = 1/x$, so IF is $e^{\int dx/x} = x$. Applying the IF gives the exact equation
$xy(y+3x+2)dx + x^2((y+x+1)dy = 0$.
$\partial F/\partial x = xy^2 + 3yx^2 + 2xy \implies F = \frac{1}{2}x^2y^2 + yx^3 + yx^2 + \phi(y)$, so the solution is $F = \frac{1}{2}x^2y^2 + yx^3 + yx^2 = C$. Since $y(1) = 2, C = 6$.

3. $y = ve^x \implies v''x - 2v' = x - 2 \implies w' - \frac{2}{x}w = 1 - \frac{2}{x}$, where $w = v'$. This linear equation gives
$w = x^2(c_1 - 1/x + 1/x^2)$, so $v = c_1x^3 - x^2/2 + x + c_2$ and $y = ve^x$.

4. (a) $f(\lambda s) = \int_0^\infty e^{-\lambda st}F dt$. Putting $v = \lambda t$ gives
$f(\lambda s) = \int_0^\infty e^{-vs}\frac{1}{\lambda}F(\frac{v}{\lambda})dv = L[\frac{1}{\lambda}f(\frac{t}{\lambda})]$.
(b) We have $\int_0^\omega e^{-st}F(t)dt = \int_0^1 3e^{-st}dt = \frac{3}{s}(1-e^{-s})$, so
$L[F] = \frac{3}{s}(1-e^{-s})/(1-e^{-2s}) = \frac{3}{s(1+e^{-s})}$.

5. (a) $L[\int_0^t F(u)du] = L[F(t) * 1] = f(s)L[1] = f(s)/s$.
(b) $L[e^{at}] = \frac{1}{s-a}, L[te^{at}] = \frac{1}{(s-a)^2}, \ldots, L[t^ne^{at}] = \frac{n!}{(s-a)^{n+1}}$. Now,
$L[\int_0^t \tau^n e^{a\tau}d\tau] = n!/s(s-a)^{n+1}$, so $L^{-1}[n!/s(s-a)^{n+1}] = \int_0^t \tau^n e^{a\tau}d\tau$.

6. $F = \cos t + \alpha(t-\pi)\cos(t-\pi), L[F] = \frac{s}{1+s^2} + \frac{se^{-\pi s}}{1+s^2}$. Put $L[y] = Y(s)$. Then $L[y''] = s^2Y \implies (1+s^2)Y = \frac{s(1+e^{-\pi s})}{1+s^2} \implies Y = \frac{s}{(1+s^2)^2} + \frac{se^{-\pi s}}{(1+s^2)^2}$.
Now, $L[\sin t] = \frac{1}{1+s^2}$, so $L[t\sin t] = \frac{2s}{(1+s^2)^2}$ and $L^{-1}[\frac{s}{(1+s^2)^2}] = \frac{1}{2}t\sin t$. Hence $L^{-1}[\frac{se^{-\pi s}}{(1+s^2)^2}] = \frac{1}{2}\alpha(t-\pi)(t-\pi)\sin(t-\pi)$, so $y = \frac{1}{2}[t\sin t + \alpha(t-\pi)(t-\pi)\sin(t-\pi)]$.

7. Put $L[y(t)] = Y(s)$. Then the integral equation gives
$Y = \frac{1}{s^2} + YL[\sin 2t] = \frac{1}{s^2} + Y\frac{2}{4+s^2}$, so $Y = \frac{1}{s^2}\frac{4+s^2}{s^2+2} = \frac{2}{s^2} - \frac{1}{s^2+2}$, hence $y = 2t - \frac{1}{\sqrt{2}}\sin\frac{t}{\sqrt{2}}$.

8. $y = \sum a_nx^n, y' = \sum na_nx^{n-1}, y'' = \sum n(n-1)a_nx^{n-2}$. Our ODE gives $2a_2 - a_0 + \sum_3^\infty[n(n-1)a_n + (n-1)(n-2)a_{n-1} - a_{n-2}]x^{n-2} = 0$. Since $a_0 = 0, a_1 = 1$ we have $a_2 = 0$ and
$a_n = \frac{a_{n-2}}{n(n-1)} - \frac{n-2}{n}a_{n-1}, n \geq 3$.
Hence $a_3 = 1/6, a_4 = -1/12, a_5 = 7/120$.