Kuwait University	**Math 240**	**Date:**	**28 November 2012**
Department of Mathematics	**2nd Midterm Exam**	**Duration:**	**90 minutes**

Calculators and mobile phones are not allowed

1. (a) (3pt)
   i. Find a homogeneous differential equation with constant coefficients satisfied by $y_1 = 1, y_2 = x$ and $y_3 = e^{2x}$.
   ii. Show that y_1, y_2, y_3 form a set of linearly independent functions over any interval.

   (b) (2pt) Find $(xD - 2)(D + 3x)(\sin x)$.

2. (5pt) Find the solution of the equation $(D^3 - 2D - 4)y = 0$, satisfying $y(0) = 1, y'(0) = 0 = y''(0)$.

3. (5pt) Use the fact that $y = x^2$ is a solution of the equation $x^2y'' - 3xy' + 4y = 0$ to find the general solution of $x^2y'' - 3xy' + 4y = x$.

4. (5pt) Use the method of undetermined coefficients to find the general solution of $(D^3 + 2D^2 - 4D - 8)y = 3e^{-x} - 4x - 6$.

5. (5pt) Use the method of variation of parameters to find a particular solution of $y'' + y = \tan^2 x$.

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1. (a) i. The roots of the auxiliary equation are $m = 0, 0, 2$, so the ODE is $D^2(D-2)y = 0$.

   ii. $W(y_1, y_2, y_3) = 4e^{2x} \neq 0$ for all x, so the y_1, y_2, y_3 are linearly independent over any interval.

   (b) $(xD-2)(D+3x)(\sin x) = (xD-2)(\cos x + 3x\sin x) =$
   $-x\sin x + x(3\sin x + 3x\cos x) - 2\cos x - 6x\sin x = 3x^2\cos x - 4x\sin x - 2\cos x$.

2. The auxiliary equation is $0 = m^3 - 2m - 4 = (m-2)((m+1)^2+1)$, so $y = ae^{2x} + be^{-x}\sin x + ce^{-x}\cos x, y' = 2ae^{2x} - (b+c)e^{-x}\sin x + (b-c)e^{-x}\cos x$, $y'' = 4ae^{2x} + (b+c)e^{-x}\sin x - (b+c)e^{-x}\cos x - (b-c)e^{-x}\cos x - (b-c)e^{-x}\sin x$. The initial conditions give $a+c = 1, 2a+b-c = 0, 4a-b-c-b+c = 0$, which leads to $a = 1/5, b = 2/5, c = 4/5$.

3. Try a solution of the form $y = x^2 v$. Then $y' = 2xv + x^2v', y'' = 2v + 4xv' + x^2v''$. Substitution into the ODE gives $x^2(2v + 4xv' + x^2v'') - 3x(2xv + x^2v) + 4x^2v = x$, which reduces to $x^3v'' + x^2v = 1$. Put $w = v'$. Then $w' + \frac{1}{x}w = \frac{1}{x^3}$, which has an integrating factor x, so $d(xw) = \frac{1}{x^2}$ and $xw = -\frac{1}{x} + c_1$ or $v' = w = -\frac{1}{x^2} + c_1/x$. This integrates to $v = \frac{1}{x} + c_1 \ln x + c_2$ and finally the general solution is $y = x + c_1x^2\ln x + c_2x^2$.

4. The auxiliary equation of the homogeneous ODE is $0 = m^3 + 2m^2 - 4m - 8 = (m-2)(m+2)^2$, so $y_c = c_1e^{2x} + c_2e^{-2x} + c_3xe^{-2x}$.

   Put $y_p = a + bx + ce^{-x}$. Then $y_p' = b - ce^{-x}, y_p'' = ce^{-x}, y_p^{(3)} = -ce^{-x}$, so $-ce^{-x} + 2ce^{-x} - 4b + 4ce^{-x} - 8a - 8bx - 8ce^{-x} = 3e^{-x} - 4x - 6$, so $a = b = 1/2, c = -1$. General solution is $y = y_c + y_p$.

5. The general solution of $y'' + y = 0$ is $A\sin x + B\cos x$ where A, B are constants, so try a particular solution of the form $y = A\sin x + B\cos x$ where A, B are functions with $A'\sin x + B'\cos x = 0$. Then $y' = A\cos x - B\sin x$, $y'' = A'\cos x - A\sin x - B'\sin x - B\cos x$. Substitution into our ODE gives $A'\cos x - B'\sin x = \tan^2 x$, so $A' = \frac{\sin^2 x}{\cos x}, B' = -\frac{\sin^3 x}{\cos^2 x}$. and $A = \int \frac{\sin^2 x}{\cos x}dx$. Substituting $c = \cos x$ gives $A = \ln(\sec x + \tan x) - \sin x$, and $B = -\sec x - \cos x$.