Kuwait University	*Math. 240*	*December 25, 2012*
Deptartment of Mathematics	*Final Examination*	*Duration 2 hours*

Calculators of any kind are not allowed. All communication devices must be switched off.

Answer the following questions. Each question or sub-question is worth 4 points.

1. Solve the following differential equations

   (a) $\left(y - e^y \sec^2 x\right) dx + (x - e^y \tan x)\, dy = 0.$

   (b) $\dfrac{dy}{dx} = \dfrac{2y}{x} + \cos\left(\dfrac{y}{x^2}\right).$

   (c) $y'' = (x + y')^2.$

2. Use the method of variation of parameters to find a particular solution of the nonhomogeneous equation

$$x^2 y'' + xy' - y = x^2 e^{-x} \ ; \ x > 0$$

   given that $y_1 = x^{-1}$ and $y_2 = x$ are solutions of the associated homogeneous equation.

3. Use the method of undetermined coefficients to solve the equation

$$y'' - 6y' + 9y = 2e^{3x} + 7 \sin 3x$$

4. Let $F(t) = \begin{cases} \sin t, & t < 2\pi \\ \sin t + \cos t, & t > 2\pi. \end{cases}$

   Express $F(t)$ in terms of $\alpha-$step function, then find the Laplace transform of $F(t)$.

5. Evaluate $L^{-1}\left\{\dfrac{(s-3)\, e^{-\pi s}}{s^2 - 4s + 8}\right\}.$

6. Given that $G(t) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^t e^{-u^2} du$. Show that $L^{-1}\left\{\dfrac{1}{(s-1)\sqrt{s}}\right\} = e^t G\left(\sqrt{t}\right).$

7. Use Laplace transform to find a nontrivial solution for the initial value problem

$$tX''(t) + (3t - 1)\, X'(t) + 3X(t) = 0, \quad X(0) = X'(0) = 0$$

8. Find the first six non-zero terms of the power series solution near the origin of the equation

$$y'' - x^2 y' - 2xy = 0$$

See backside for formulas

TABLE FOR LAPLACE TRANSFORMS

$F(t)$	$f(s)$	$F(t)$	$f(s)$
1	$\frac{1}{s}$	$\sin bt$	$\frac{b}{s^2+b^2}$
e^{at}	$\frac{1}{s-a}$	$\cos bt$	$\frac{s}{s^2+b^2}$
t^n	$\frac{n!}{s^{n+1}}$	$\sinh bt$	$\frac{b}{s^2-b^2}$
t^x	$\frac{\Gamma(x+1)}{s^{x+1}}$	$\cosh bt$	$\frac{s}{s^2-b^2}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$		

FORMULAS FOR LAPLACE TRANSFORMS

$$L\{F(t)\} = f(s)$$

$$L\{t^n F(t)\} = (-1)^n f^{(n)}(s)$$

$$L\{F'(t)\} = sf(s) - F(0)$$

$$L\{e^{at}F(t)\} = f(s-a)$$

$$L\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)$$

$$L\{F(t-c)\,\alpha(t-c)\} = e^{-cs} f(s)$$

$$L\{F'''(t)\} = s^3 f(s) - s^2 F(0) - sF'(0) - F'(0)$$

$$L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)\,du$$

$$L\left\{\int_0^t F(u)\,G(t-u)\,du\right\} = f(s)\,g(s)$$

$$L\left\{\int_0^t F(u)\,du\right\} = \frac{f(s)}{s}$$

1. a $(ydx + xdy) - (e^y d\tan x + \tan x de^y) = 0 \implies d(xy - e^y \tan x) = 0 \implies xy - e^y \tan x = c.$

1. b Put $y = vx^2 \implies x^2 v' = \cos v \implies \sec v dv = x^{-2} dx \implies \ln|\sec v + \tan v| + x^{-1} = c.$

1. c. Put $v = x + y' \implies v' = 1 + v^2 \implies \dfrac{dv}{1+v^2} = dx \implies \tan^{-1} v = x + c_1$

$$v = x + y' = \tan(x + c_1) \implies y' = \tan(x + c_1) - x \implies y = \ln|\sec(x + c_1)| - \tfrac{1}{2}x^2 + c_2$$

2. $y_c = c_1 x^{-1} + c_2 x \implies y_p = u_1 x^{-1} + u_2 x \implies u_1' x^{-1} + u_2' x = 0$ and $u_1'\left(-x^{-2}\right) + u_2' = e^{-x}$

$$u_1 = -\tfrac{1}{2}\int x^2 e^{-x} dx = \left(\tfrac{1}{2}x^2 + x + 1\right)e^{-x} \text{ and } u_2 = \tfrac{1}{2}\int e^{-x} dx = -\tfrac{1}{2}e^{-x} \implies y_p = \left(1 + x^{-1}\right)e^{-x}.$$

3. $y_c = (c_1 + c_2 x)e^{3x}$ and $y_p = Ax^2 e^{3x} + B\sin 3x + C\cos 3x = x^2 e^{3x} + \frac{7}{18}\cos 3x.$

4. $F(t) = \sin t + \cos(t - 2\pi)\,\alpha(t - 2\pi) \implies L\{F(t)\} = \dfrac{1}{s^2+1} + \dfrac{s}{s^2+1}e^{-2\pi s}.$

5. $L^{-1}\left\{\dfrac{(s-3)e^{-\pi s}}{s^2 - 4s + 8}\right\} = F(t - \pi)\,\alpha(t - \pi).$

$$F(t) = L^{-1}\left\{\frac{s-3}{s^2-4s+8}\right\} = L^{-1}\left\{\frac{(s-2)-1}{(s-2)^2+4}\right\} = e^{2t}\left(\cos 2t - \tfrac{1}{2}\sin 2t\right).$$

6. $L^{-1}\left\{\dfrac{1}{(s-1)\sqrt{s}}\right\} = \displaystyle\int_0^t e^{t-\beta}\frac{1}{\sqrt{\pi\beta}}d\beta = \frac{e^t}{\sqrt{\pi}}\int_0^t \frac{e^{-\beta}}{\sqrt{\beta}}d\beta : \xrightarrow{\beta = u^2} = \frac{2e^t}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}du = e^t G\left(\sqrt{t}\right).$

7. Let $L\{X(t)\} = Y(s) \implies L\{X'(t)\} = sY(s)$ and $L\{X''(t)\} = s^2 Y(s)$

$$L\{tX''\} + 3L\{tX'\} - L\{X'\} + L\{X\} = 0 \implies -\left[s^2 Y(s)\right]' - 3[sY(s)]' - [sY(s)] + Y(s) = 0$$

$$\frac{Y'(s)}{Y(s)} = -\frac{3}{s+3} \implies \ln Y(s) = \ln c(s+3)^{-3} \implies Y(s) = \frac{c}{(s+3)^3}$$

$$X(t) = L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{c}{(s+3)^3}\right\} = kt^2 e^{-3t}\ ;\ k \neq 0.$$

8. $y = \sum\limits_{n=0} a_n x^n \implies y' = \sum\limits_{n=1} n a_n x^{n-1}$ and $y'' = \sum\limits_{n=2} (n-1) n a_n x^{n-2}.$

$$\sum_{n=2} (n-1) n a_n x^{n-2} - \sum_{n=1} n a_n x^{n+1} - 2\sum_{n=0} a_n x^{n+1} = 0$$

$$\sum_{n=0} (n+1)(n+2) a_{n+2} x^n - \sum_{n=1} (n+1) a_{n-1} x^n = 0$$

$$2a_2 + \sum_{n=1} \left[(n+1)(n+2)a_{n+2} - (n+1)a_{n-1}\right]x^n = 0 \implies a_2 = 0 \text{ and } a_{n+2} = \frac{a_{n-1}}{n+2};\ n \geq 1.$$

$$y = a_0\left(1 + \tfrac{1}{3}x^3 + \tfrac{1}{18}x^6 + \cdots\right) + a_1\left(x + \tfrac{1}{4}x^4 + \tfrac{1}{28}x^7 + \cdots\right).$$