MATHEMATICS DEPARTMENT **KUWAIT UNIVERSITY**

Differential equations Math-240

Date: July 9, 2012 **Second Examination** **Duration : 90 Minutes**

Answer all questions Calculators and mobile phones are not allowed

1. **a.** If the functions y_1 and y_2 are linearly independent, determine under what conditions the functions $f = a_1y_1 + a_2y_2$ and $g = b_1y_1 + b_2y_2$ (where $a_1, a_2, b_1,$ and b_2 are constants) also form a linearly independent set of functions.

b. Let $y_1 = x$ and $y_2 = e^x$ are linearly independent solutions of the equation,

$$y'' + \frac{x}{1-x}y' - \frac{1}{1-x}y = 0, \qquad x \neq 1 . \quad \cdots\cdots\cdots \ (2)$$ **4**

Verify that $f = x + 3e^x$ and $g = 2x - e^x$ are solutions of equation (2) and show that

$$W(f,g) = -7W(y_1,y_2) = -7W(x,e^x).$$

2. **a.** Given $f(D) = xD - 1$ and $g(D) = 2D + x$ are two differential operators and C_1 and C_2 are arbitrary constants. Show that the product $(f(D)g(D))$ is a linear differential operator.

b. Show that $y_1 = 1$ and $y_2 = x^{\frac{1}{2}}$ are solutions of the equation,

$$yy'' + (y')^2 = 0,$$ **4**

but that their sum $y = y_1 + y_2$ is not a solution.

3. Solve the given differential equation subject to indicated boundary conditions,

$$y''' - y'' + 2y = 0, \quad y(0) = 2,\ y'(0) = 0,\ y''(0) = 1 .$$ **4**

4. Use the method of **undetermined coefficients** and the principle of superposition to find the general solution of the equation,

$$y'' + 4y' + 4y = 27xe^x + 25\cos x.$$ **5**

5. The differential equation

$$y'' + a(xy' + y) = 0, \quad a \text{ is a constant },$$ **4**

arises in the study of turbulent flow in the wake of a cylinder.

a. Verify that $y_1(x) = e^{-ax^2/2}$ is a solution of this equation.

b. Find the general solution in the form of an integral.

6. Find the general solution to the following equation by a method of **variation of parameters**,

$$3y'' + 2y' = 26\sin x.$$ **4**

Math-240
second mid-term

July 9, 2012

Answer

1-a) $W(y_1,y_2) = (y_1y_2' - y_2y_1') \neq 0$ solutions are linearly independent

$$W(f,g) = \begin{vmatrix} a_1y_1 + a_2y_2 & b_1y_1 + b_2y_2 \\ a_1y_1' + a_2y_2' & b_1y_1' + b_2y_2' \end{vmatrix} = (a_1b_2 - a_2b_1)(y_1y_2' - y_2y_1') = (a_1b_2 - a_2b_1)W(y_1,y_2)$$

The condition is $a_1b_2 - a_2b_1 \neq 0$.

1-b) $\left(x + 3e^x\right)'' + \frac{x}{1-x}\left(x + 3e^x\right)' - \frac{1}{1-x}\left(x + 3e^x\right) = 3e^x + \frac{x}{1-x}\left(1 + 3e^x\right) - \frac{1}{1-x}\left(x + 3e^x\right) = 0$

$\left(2x - e^x\right)'' + \frac{x}{1-x}\left(2x - e^x\right)' - \frac{1}{1-x}\left(2x - e^x\right) = -e^x + \frac{x}{1-x}\left(2 - e^x\right) - \frac{1}{1-x}\left(2x - e^x\right) = 0$

$$W(y_1,y_2) = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = e^x(x-1) \neq 0, \quad W(f,g) = \begin{vmatrix} x + 3e^x & 2x - e^x \\ 1 + 3e^x & 2 - e^x \end{vmatrix} = -7e^x(x-1) = -7\,W(y_1,y_2)$$

2-a) $f(D) = xD - 1$ and $g(D) = 2D + x$

So that product is $f(D)g(D)y = (xD-1)(2D+x)y = (xD-1)(2Dy + xy) = 2xD^2y + (x^2-2)Dy$

$f(D)g\,(D)(C_1Y_1 + C_2Y_2) = (2xD^2 + (x^2-2)D)(C_1Y_1 + C_2Y_2)$

$= C_1(2xD^2 + (x^2-2)D)Y_1 + C_2(2xD^2 + (x^2-2)D)Y_2 = C_1f(D)g\,(D)Y_1 + C_2f(D)g\,(D)Y_2$

2-b) $y_1 = 1$ and $y_2 = x^{\frac{1}{2}}$ are solutions of the equation, $yy'' + (y')^2 = 0$,

$(1)(1)'' + \left((1)'\right)^2 = 0 + 0 = 0,\qquad \left(x^{\frac{1}{2}}\right)\left(x^{\frac{1}{2}}\right)'' + \left(\left(x^{\frac{1}{2}}\right)'\right)^2 = \left(x^{\frac{1}{2}}\right)\left(-\frac{1}{4}x^{-\frac{3}{2}}\right) + \left(\frac{1}{2}x^{-\frac{1}{2}}\right)^2 = 0$

but that their sum $y = 1 + x^{\frac{1}{2}}$,

$\left(1 + x^{\frac{1}{2}}\right)\left(1 + x^{\frac{1}{2}}\right)'' + \left(\left(1 + x^{\frac{1}{2}}\right)'\right)^2 = \left(1 + x^{\frac{1}{2}}\right)\left(-\frac{1}{4}x^{-\frac{3}{2}}\right) + \left(\frac{1}{2}x^{-\frac{1}{2}}\right)^2 = -\frac{1}{4}x^{-\frac{3}{2}} \neq 0$

So that the sum $y = 1 + x^{\frac{1}{2}}$ is not a solution .

3) $y''' - y'' + 2y = 0, \quad y(0) = 2,\ y'(0) = 0,\ y''(0) = 1$.

Auxiliary equation is $m^3 - m^2 + 2 = (m+1)(m^2 - 2m + 2) = 0,\ m = -1,\ m = 1 \pm i$

The general solution is

$y_g = c_1e^{-x} + e^x(c_2\cos x + c_3\sin x) \qquad \Rightarrow y_g(0) = 2 = c_1 + c_2$

$y_g' = -c_1e^{-x} + e^x[c_2(\cos x - \sin x) + c_3(\sin x + \cos x)] \qquad \Rightarrow y_g'(0) = 0 = -c_1 + c_2 + c_3$

$y_g'' = c_1e^{-x} + e^x[-2c_2\sin x + 2c_3\cos x] \qquad \Rightarrow y_g''(0) = 1 = c_1 + 2c_3$

$c_1 = 1,\ c_2 = 1,\ c_3 = 0$ So the particular solution is $y_g = e^{-x} + e^x\cos x$

4) Using undetermined coefficients to solve the equation

$y'' + 4y' + 4y = 27xe^x + 25\cos x$, Homogeneous equation $y'' + 4y' + 4y = 0$

Auxiliary equation $m^2 + 4m + 4 = (m+2)^2 = 0$, Roots $(-2,-2)$ and $y_c = c_1e^{-2x} + c_2xe^{-2x}$

To find y_p the first term xe^x obtain from root $m' = 1,\ 1$ and operator $(D-1)^2$ so that $y_{p_1} = Ae^x + Bxe^x$

and second term $\cos x$ obtain from root $m' = \pm i$ and operator $D^2 + 1$ so that $y_{p_2} = C\cos x + E\sin x$

Substitute in the equation $(Ae^x + Bxe^x)'' + 4(Ae^x + Bxe^x)' + 4(Ae^x + Bxe^x) = 27xe^x$

$e^x(A + 2B + Bx) + 4e^x(A + B + Bx) + 4e^x(A + Bx) = 27xe^x \Rightarrow B = 3,\quad 9A + 6B = 0,\ A = -2$

So $y_{p_1} = -2e^x + 3xe^x$

and $(C\cos x + E\sin x)'' + 4(C\cos x + E\sin x)' + 4(C\cos x + E\sin x) = 25\cos x$

$(-C\cos x - E\sin x) + 4(-C\sin x + E\cos x)' + 4(C\cos x + E\sin x) = 25\cos x \Rightarrow$

$4E + 3C = 25,\ -4C + 3E = 0,\quad C = 3\ ,\quad E = 4,\quad y_{p_2} = 3\cos x + 4\sin x$

So that $y_p = y_{p_1} + y_{p_2} = -2e^x + 3xe^x + 3\cos x + 4\sin x$

5) $y'' + a(xy' + y) = 0$, a is a constant , and $y_1(x) = e^{-ax^2/2}$

a) $\left(e^{-ax^2/2}\right)'' + a\left(x\left(e^{-ax^2/2}\right)' + e^{-ax^2/2}\right) = e^{-ax^2/2}(a^2x^2 - a) + a\left[xe^{-ax^2/2}(-ax) + e^{-ax^2/2}\right]$

$$= e^{-ax^2/2}[a^2x^2 - a - a^2x^2 + a] = 0$$

b) To find the general solution let

$$y_g = vy_1 = ve^{-ax^2/2},$$
$$y_g' = v'e^{-ax^2/2} - axve^{-ax^2/2} = e^{-ax^2/2}(v' - axv)$$
$$y_g'' = e^{-ax^2/2}(v'' - 2axv' + (a^2x^2v - av))$$

Substitute $e^{-ax^2/2}(v'' - 2axv' + v(a^2x^2 - a)) + a\left[xe^{-ax^2/2}(v' - axv) + ve^{-ax^2/2}\right] = 0$

or $(v'' - 2axv' + v(a^2x^2 - a)) + a(x(v' - axv) + v) = 0$

or $v'' - axv' = 0$, Let $w = v'$, $w' = v''$ So that $w' - axw = 0$,

$\frac{1}{w}dw - axdx = 0$, $\Rightarrow \ln|w| = \frac{1}{2}ax^2 + \ln C$, $\Rightarrow w = Ce^{\frac{1}{2}ax^2} = v'$

So that $v = \int Ce^{\frac{1}{2}ax^2}dx + C_1$, Finally $y_g = ve^{-ax^2/2} = e^{-ax^2/2}\left(\int Ce^{\frac{1}{2}ax^2}dx + C_1\right)$

6) Using variation of parameter $3y'' + 2y' = 26\sin x$

$$y_c = C_1 + C_2e^{-\frac{2}{3}x}$$

Let $y_p = A(x) + B(x)e^{-\frac{2}{3}x}$, $y_p' = A' + B'e^{-\frac{2}{3}x} - \frac{2}{3}Be^{-\frac{2}{3}x}$, Set $A' + B'e^{-\frac{2}{3}x} = 0$

$$y_p' = -\frac{2}{3}Be^{-\frac{2}{3}x}, \quad y_p'' = -\frac{2}{3}B'e^{-\frac{2}{3}x} + \frac{4}{9}Be^{-\frac{2}{3}x}$$

Substitute $3\left(-\frac{2}{3}B'e^{-\frac{2}{3}x} + \frac{4}{9}Be^{-\frac{2}{3}x}\right) + 2\left(-\frac{2}{3}Be^{-\frac{2}{3}x}\right) = -2B'e^{-\frac{2}{3}x} = 26\sin x$

$B' = -13e^{\frac{2}{3}x}\sin x$ and $A' = -B'e^{-\frac{2}{3}x} = 13e^{\frac{2}{3}x}e^{-\frac{2}{3}x}\sin x = 13\sin x$, $\boxed{A = -13\cos x}$

$$B = -\int 13e^{\frac{2}{3}x}\sin x dx = 9(\cos x)e^{\frac{2}{3}x} - 6(\sin x)e^{\frac{2}{3}x}$$
$$y_p = A(x) + B(x)e^{-\frac{2}{3}x} = -13\cos x + \left(9(\cos x)e^{\frac{2}{3}x} - 6(\sin x)e^{\frac{2}{3}x}\right)e^{-\frac{2}{3}x}$$
$$= -13\cos x + 9(\cos x) - 6(\sin x) = -4\cos x - 6\sin x$$
$$y_g = C_1 + C_2e^{-\frac{2}{3}x} + -4\cos x - 6\sin x$$