Kuwait University
Department of Mathematics

Math 240 — June 26, 2012

Intro. Ord. Diff. Eqns. — MidTerm 1 — Time: 90 minutes

---

1. Employ <u>two different methods</u> to find the general solution of the equation [7 pts]

$$y' = (y + y^{-3})\cos x$$

then determine the solution curve that passes through the point $P(0, -1)$.

2. Find the general solution of the equation [3 pts]

$$\frac{dy}{dx} = \frac{x + y - 4}{x - 4y - 4}$$

3. Solve the equation [3 pts]

$$\left(\frac{1}{x^2} - \cos y\right) dx + \left(x \sin y - \frac{1}{y}\right) dy = 0$$

4. (a) Find the general solution of the equation [3 pts]

$$(y^2 + 2y - x^2)\, dx + x\,(y + 1)\, dy = 0$$

(b) Find the orthogonal trajectories for the family of curves defined by [3 pts]

$$x^2 + y^2 + 2y = c\,x$$

5. To find the general solution of the equation $x\,y' + 2y = x\,e^{x^3}$, which of the following procedures is suitable to start with? Only one of them is valid. [3 pts]

   (a) make the change of variable x to $t = e^{x^3}$

   (b) multiply both sides of the equation by $3x^2$

   (c) multiply both sides of the equation by x^2

   Find the general solution based on your choice of procedure.

6. To find the general solution of the equation [3 pts]

$$\frac{dy}{dx} = \frac{2y^2}{x^2 + y^2}$$

which of the following substitutions is suitable to start with? Only one of them is valid.

   (a) Put $x = v\,y$ to get a separable equation

   (b) Put $v = x\,y$ to get a Bernoulli equation

   Find the general solution based on your choice above.

S O L U T I O N S

1. As a separable equation:

$$\frac{y^3}{1+y^4}\,dy = \cos x\,dx \quad\to\quad \ln(1+y^4) = 4\,\sin x + c$$

As a Bernoulli equation:

$$y' - \cos x\,y = y^{-3}\,\cos x \quad\to\quad z = y^{1-(-3)} = y^4 \quad\to\quad y^3\,y' - \cos x\,y^4 = \cos x \quad\to$$

$$\frac{1}{4}\,z' - \cos x\,z = \cos x \quad\to\quad z' - 4(1+z)\,\cos x = 0 \quad\to\quad \ln(1+z) - 4\sin x = c \quad\to$$

$$\ln(1+y^4) = 4\,\sin x + c$$

For the particular solution curve we have

$$\ln 2 = c \quad\to\quad \ln(1+y^4) - 4\sin x = \ln 2$$

2. The equation is expressed in the form

$$(x+y-4)\,dx - (x-4y-4)\,dy = 0$$

Since the two lines intersect at $(4,0)$, we put $x = u+4$ and $y = v$ so that

$$(u+v)\,du - (u-4v)\,dv = 0$$

Let $u = z\,v$ to get the separable equation

$$(z^2+4)\,dv + v\,(z+1)\,dz = 0 \quad\to\quad \frac{dv}{v} + \frac{z+1}{z^2+4}\,dz = 0 \quad\to\quad \ln\left(v^2\,(z^2+4)\right) + \frac{1}{2}\,\tan^{-1}\frac{z}{2} = c$$

The solution family becomes

$$\ln(u^2+4v^2) + \tan^{-1}\frac{u}{2v} = c\,, \qquad u = x-4\,,\; v = y$$

3. The equation is exact and the family of solutions is given by

$$\frac{1}{x} + x\,\cos y + \ln y = c$$

4. (a)

$$\underbrace{(y^2+2y-x^2)}_{M}\,dx + \underbrace{x\,(y+1)}_{N}\,dy = 0$$

Note that this equation is not exact since $M_y = 2\,(y+1)$ and $N_x = y+1$, but

$$\frac{1}{N}\,(M_y - N_x) = \frac{1}{x} \quad\to\quad \rho = \mathrm{e}^{\int x^{-1}\,dx} = x$$

Multiply the equation by the integrating factor x to get the exact equation

$$dF = (-x^3 + x\,y^2 + 2x\,y)\,dx + x^2\,(y+1)\,dy = 0 \quad\to$$

$$F_x = -x^3 + x\,y^2 + 2x\,y \quad\to\quad F = -\frac{1}{4}\,x^4 + \frac{1}{2}\,x^2\,y^2 + x^2\,y + g(y) \quad\to\quad F_y = x^2\,y + x^2 + g'(y)$$

Put $g(y) = 0$ to get the family of solutions $x^4 - 2x^2\,y^2 - 4x^2\,y = k$.

(b) For the given family $x^2 + y^2 + 2y = c\,x$, we have

$$2x + 2(y+1)\,y' = c = \frac{x^2+y^2+2y}{x} \quad\to\quad y' = \frac{-x^2+y^2+2y}{2x\,(y+1)}$$

For the orthogonal trajectories:

$$-\frac{dx}{dy} = \frac{-x^2+y^2+2y}{2x\,(y+1)} \quad \to \quad \underbrace{2x\,(y+1)}_{M}\,dx + \underbrace{(-x^2+y^2+2y)}_{N}\,dy = 0$$

Note that $M_y = 2x$ and $N_x = -2x$. Thus,

$$M_y - N_x = 4x \quad \to \quad \frac{1}{M}\,(M_y - N_x) = \frac{2}{y+1} \quad \to \quad IF\ \rho(y) = \frac{1}{(y+1)^2}$$

Multiply the OT equation by $\rho(y)$ to get the exact equation

$$\frac{2x}{y+1}\,dx + \frac{-x^2+y^2+2y}{(y+1)^2}\,dy = 0$$

$$F_x = \frac{2x}{y+1} \quad \to \quad F = \frac{x^2}{y+1} + g(y) \quad \to \quad F_y = \frac{-x^2}{(y+1)^2} + g'(y) \quad \to$$

$$g'(y) = \frac{y^2+2y}{(y+1)^2} = \frac{(y+1)^2-1}{(y+1)^2} = 1-(y+1)^{-2} \quad \to \quad g(y) = y + (y+1)^{-1}$$

The family of orthogonal trajectories is given by

$$\frac{x^2+1}{y+1} + y = k$$

Note that if (b) were chosen, the equation needs to be divided by 3 , which takes us back to (c).

5. Write the equation in the standard form

$$y' + \frac{2}{x}\,y = \mathrm{e}^{x^3} \quad \to \quad \rho = \mathrm{e}^{2\int dx/x} = x^2$$

so the third procedure (c) is to be used, giving rise to the solution

$$x^2\,y = \int x^2\,\mathrm{e}^{x^3} = \frac{1}{3}\,\mathrm{e}^{x^3} + c_1 \quad \to \quad 3x^2\,y - \mathrm{e}^{x^3} = c$$

6. The first one (a) works since we have homogeneous coefficients of degree 2:
$x = v\,y \ \to\ dx = v\,dy + y\,dv$

$$(x^2+y^2)\,dy - 2y^2\,dx = y^2\,(v^2+1)\,dy - 2y^2\,(v\,dy + y\,dv) = 0 \quad \to \quad (v^2-2v+1)\,dy - 2y\,dv = 0 \quad \to$$

$$\frac{dy}{y} - \frac{2}{(v-1)^2}\,dv = 0 \quad \to \quad \ln y + \frac{2}{v-1} = c \quad \to \quad \ln y + \frac{2y}{x-y} = c$$