Kuwait University	**MATH 240**	July, 31, 2012
Department of Mathematics	**Final Exam**	Duration: two hours

Calculators and mobile telephones are not allowed.

Answer the following questions.

1. (4 pts) Solve the equation $2y' = \dfrac{\sin x}{ye^x} - y.$

2. (4 pts each) Find the general solution of the equations

(a) $e^{xy}y^2dx + (e^{xy}xy + y\ln y + y)dy = 0.$

(b) $(x^2y^2 + y)dx + x(2x^2y - 1)dy = 0.$

3. (4 pts) Solve the equation by the method of undetermined coefficients

$y''' + y'' - y' - y = x + 4e^x.$

4. (4 pts) Use the method of variation of parameters to find the general solution of the equation

$y'' + y = \tan x \sec x.$

5. (4 pts) Prove that if $L(F(t)) = f(s)$ and $\dfrac{F(t)}{t}$ is of class A, then

$L(\dfrac{F(t)}{t}) = \displaystyle\int_s^\infty f(x)dx.$

6. (4 pts) Find the inverse Laplace transform L^{-1} of the following functions:

(a) $\dfrac{1}{s(s^2+1)},$ (b) $e^{-5s}(\dfrac{1}{s^2} + \dfrac{2}{s}).$

7. (4 pts) Use the Laplace transform to solve the initial value problem :

$y'' + 2y' + y = te^{-2t}, \quad y(0) = 1, \quad y'(0) = 0.$

8. (4 pts) Solve the integral equation

$y(t) = 6t + 4\displaystyle\int_0^t (t-\beta)^2 y(\beta)d\beta.$

9. (4 pts) Find the first four non-zero terms of the power series solution of the initial value problem:

$y'' - xy' - y = 0, \quad y(0) = 1, \quad y'(0) = 0.$

MATH 240

FINALEXAM. July 31, 2012

SOLUTIONS

1. $2y' = \frac{\sin x}{ye^x} - y \Leftrightarrow 2yy'e^x + y^2e^x = \sin x \Leftrightarrow \left(y^2e^x\right)' = \sin x$

Hence the solution is $y^2e^x = -\cos x + C$.

2. (a)

$u = e^{xy}, du = ye^{xy}dx + xe^{xy}dy$

$ydu + y(1+\ln y)dy = 0 \Leftrightarrow du + (1+\ln y)dy = 0$

$u + \int(1+\ln y)dy = C \Rightarrow e^{xy} + y\ln y = C$

(b)

$M(x,y) = x^2y^2 + y, N(x,y) = x\left(2x^2y - 1\right)$

$\frac{\partial M}{\partial y} = 2x^2y + 1, \frac{\partial N}{\partial x} = 6x^2y - 1$

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -4x^2y + 2 \Rightarrow \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{2}{x}$

Then the integrating factor takes the form $\exp(-\int \frac{2}{x}dx) = \frac{1}{x^2}$ and thus the equation

$$\left(y^2 + \frac{x}{y^2}\right)dx + \left(2xy - \frac{1}{x}\right)dy = 0$$

is an exact one.

$$\frac{\partial F}{\partial x} = y^2 + \frac{x}{y^2} \Rightarrow F(x,y) = xy^2 - \frac{y}{x} + g(y)$$

$$\frac{\partial F}{\partial y} = 2xy - \frac{1}{x} \Rightarrow g'(y) = 0 \Rightarrow g(y) = C$$

Then the general solution is given by $xy^2 - \frac{y}{x} = C$.

3.

$$y''' + y'' - y' - y = x + 4e^{x}$$

$$y'''+y''-y'-y=0 \Leftrightarrow (D-1)(D+1)^2 y=0 \Rightarrow y_c=C_1e^{x}+C_2e^{-x}+C_3xe^{-x}$$

$$y_p = A + Bx + Cxe^{x}$$

$$y'_p=B+Ce^{x}+Cxe^{x}, y''_p=2Ce^{x}+Cxe^{x}, y'''_p=3Ce^{x}+Cxe^{x}$$

$$3Ce^{x}+Cxe^{x}+2Ce^{x}+Cxe^{x}-B-Ce^{x}-Cxe^{x}-A-Bx-Cxe^{x}=x+4e^{x}$$

$$-A-B-Bx+4Ce^{x}=x+4e^{x} \Rightarrow A=1, B=-1, C=1 \Rightarrow y_p=1-x+xe^{x}$$

Then the general solution is

$$y = C_1e^{x} + C_2e^{-x} + C_3xe^{-x} + 1 - x + xe^{x}$$

4.

$$y'' + y = \frac{\sin x}{\cos^2 x}$$

$$y_p=A(x)\cos x+B(x)\sin x$$

$$A'\cos x+B'\sin x=0$$

$$-A'\sin x+B'\cos x=\frac{\sin x}{\cos^2 x}$$

$$B'=\tan x \Rightarrow B=-\ln|\cos x|$$

$$A'=-\frac{\sin^2 x}{\cos^2 x} \Rightarrow A=x-\tan x$$

$$y_p=x\cos x-\sin x-\sin x\ln|\cos x| \Rightarrow y_p=x\cos x-\sin x\ln e|\cos x|$$

Hence the general solution of the equation has the form

$$y = C_1 \cos x + C_2 \sin x + x\cos x - \sin x \ln e|\cos x|$$

5.

$$\int_s^\infty f(x)dx=\int_s^\infty\int_0^\infty e^{-xt}F(t)dtdx=\int_0^\infty F(t)\int_s^\infty e^{-xt}dxdt=\int_0^\infty F(t)\left[-\frac{1}{t}e^{-xt}\right]_{x=s}^{\infty}dt=\int_0^\infty e^{-xt}\frac{F(t)}{t}dt$$

6.

(a) By the Convolution Theorem:

$$L^{-1}\left[\frac{1}{s(s^2+1)}\right]=\int_0^t 1.\sin(t-\tau)d\tau=\int_0^t \sin\tau d\tau=1-\cos t$$

(b) By the shift property $L^{-1}\left[e^{-as}f(s)\right]=\alpha(t-a)F(t-a)$:

$$L^{-1}\left[e^{-5s}\left(\frac{1}{s^2}+\frac{2}{s}\right)\right]=\alpha(t-5)(t-5)+2\alpha(t-5)=\alpha(t-5)(t-3)=\begin{cases}0,0<t<5\\ t-3,t>5\end{cases}$$

7. By the transform of derivative and differentiation of transform properties:

$$L[y]=\frac{s+2}{(s+1)^2}+\frac{1}{(s+1)^2(s+2)^2}$$ or

$$L[y]=\frac{1}{s+1}+\frac{1}{(s+1)^2}+\frac{1}{(s+1)^2(s+2)^2}$$

By partial fractions decomposition:

$$\frac{1}{(s+1)^2(s+2)^2}=-\frac{2}{s+1}+\frac{1}{(s+1)^2}+\frac{2}{(s+2)}+\frac{1}{(s+2)^2}$$

Thus

$$L[y]=-\frac{1}{s+1}+\frac{2}{s+2}+\frac{2}{(s+1)^2}+\frac{1}{(s+2)^2}$$

and hence

$$y=-e^{-t}+2e^{-2t}+2te^{-t}+te^{-2t}$$ or

$$y=(2t-1)e^{-t}+(t+2)e^{-2t}$$

8. If $f(s) = L[y(t)]$, by the Convolution Theorem:

$$f(s) = \frac{6}{s^2} + \frac{8}{s^3} f(s) \Leftrightarrow f(s) = \frac{6s}{(s-2)(s^2+2s+4)}$$

The partial fractions decompositions leads to

$f(s) = \frac{1}{s-2} - \frac{s-2}{s^2+2s+4}$ or equivalently $f(s) = \frac{1}{s-2} - \frac{(s+1)-3}{(s+1)^2+3}$.

Therefore

$$y(t) = e^{2t} - L^{-1}\left[\frac{(s+1)-3}{(s+1)^2+3}\right]$$

The shift property $L^{-1}[f(s)] = e^{-at}L^{-1}\left[f(s-a)\right]$ gives then

$$y(t) = e^{2t} - e^{-t}\left\{L^{-1}\left[\frac{s}{s^2+3}\right] - \sqrt{3}L^{-1}\left[\frac{\sqrt{3}}{s^2+3}\right]\right\},$$

and hence

$$y(t) = e^{2t} - e^{-t}(\cos\sqrt{3}t - \sqrt{3}\sin\sqrt{3}t)$$

9.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, a_0 = 1, a_1 = 0$$

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}, y''(x) = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$$

The substitution of y, y', y'' into the equation implies

$$\sum_{n=0}^{\infty} [n(n-1)a_n - (n-1)a_{n-2}]x^{n-2} = 0$$

and thus we obtain the recurrence formula for the coefficients $a_n = \frac{1}{n}a_{n-2}, n \geq 2$.

From the initial conditions it follows readily that

$a_{2k} = \frac{1}{2^k k!}$ and $a_{2k+1} = 0$ for $k \geq 0$. Then the first four non-zero terms of the power series solution are 1, $\frac{x^2}{2}$, $\frac{x^4}{2^2.2!}$ and $\frac{x^6}{2^3.3!}$.