Kuwait University — Department of Mathematics

Math 240 : Ordinary Differential Equations

Spring Semester 2012 — Second In-Term Test — Sunday 22 April 2012

Duration: 90 minutes.
Calculators and electronic communication devices are NOT allowed.
Read each question carefully, and show all your work in order to receive full credit.
Total marks: 25.

1. [3 marks] Show that the functions $f_1(x) = 1$, $f_2(x) = |x-1|$, and $f_3(x) = |x+1|$ are linearly independent on $(-\infty, \infty)$.

2. [3 marks] Suppose that f_1, f_2, and g are differentiable functions. Let W be the Wronskian of f_1 and f_2. Show that the Wronskian of gf_1 and gf_2 is equal to g^2W.

3. [3 marks] Let $A = (\cosh x)D$ and $B = (\sinh x)D$ where D denotes differentiation with respect to x. Find the differential operator $AB - BA$.

4. [5 marks] Find the general solution of

$$y''' + 4y'' + y' + 4y = 5e^{2x} + 34\sin x.$$

5. [5 marks] Given that $y = x$ is a solution, use the method of reduction of order to find the general solution of

$$x^3y''' - 3x^2y'' + x\left(6 - x^2\right)y' - \left(6 - x^2\right)y = 0$$

6. (a) [1 mark] Verify that $y_1 = x$ and $y_2 = xe^x$ are solutions of

$$x^2y'' - x(x+2)y' + (x+2)y = 0.$$

   (b) [1 mark] Verify that y_1 and y_2 are linearly independent.

   (c) [4 marks] Use the method of variation of parameters to find the general solution of

$$x^2y'' - x(x+2)y' + (x+2)y = 2x^3.$$

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MODEL ANSWERS

1. The functions are linearly independent if

$$c_1(1) + c_2|x-1| + c_3|x+1| = 0$$

for some constants c_1, c_2 and c_3 necessarily implies that $c_1 = c_2 = c_3 = 0$. Substituting $x = 1$ and $x = -1$ in the above equation gives $c_1 + 2c_3 = 0$ and $c_1 + 2c_2 = 0$ respectively. Hence, $c_2 = c_3 = -c_1/2$, and the above equation becomes $c_1(1 - |x-1|/2 - |x+1|/2) = 0$. Passing to the limit $x \to \pm\infty$ gives $c_1 = 0$. Thus

$$c_1 = c_2 = c_3 = 0.$$

2. The Wronskian of gf_1 and gf_2 is

$$\begin{aligned} \begin{vmatrix} gf_1 & gf_2 \\ (gf_1)' & (gf_2)' \end{vmatrix} &= \begin{vmatrix} gf_1 & gf_2 \\ gf_1' + g'f_1 & gf_2' + g'f_2 \end{vmatrix} = g \begin{vmatrix} f_1 & f_2 \\ gf_1' + g'f_1 & gf_2' + g'f_2 \end{vmatrix} \\ &= g \begin{vmatrix} f_1 & f_2 \\ gf_1' & gf_2' \end{vmatrix} = g^2 \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = g^2 W. \end{aligned}$$

3. For any twice differentiable function y,

$$\begin{aligned} (AB - BA)\,y &= (\cosh x)\left[(\sinh x)\,y'\right]' - (\sinh x)\left[(\cosh x)\,y'\right]' \\ &= (\cosh x)\left[(\cosh x)\,y' + (\sinh x)\,y''\right] \\ &\quad - (\sinh x)\left[(\sinh x)\,y' + (\cosh x)\,y''\right] \\ &= \left(\cosh^2 x - \sinh^2 x\right) y' = y'. \end{aligned}$$

Answer: $AB - BA = D$.

4. The auxiliary equation is

$$0 = m^3 + 4m^2 + m + 4 = (m+4)\left(m^2+1\right) = (m+4)(m-i)(m+i).$$

Hence, the complementary function is

$$y_c = c_1 e^{-4x} + c_2 \cos x + c_3 \sin x$$

where c_1, c_2 and c_3 are arbitrary constants.

Using the method of undetermined coefficients, substitute

$$y = Ae^{2x} + Bx\cos x + Cx\sin x$$

with A, B and C unknown numbers. This gives

$$y' = 2Ae^{2x} + B\cos x - Bx\sin x + C\sin x + Cx\cos x,$$

$$y'' = 4Ae^{2x} - 2B\sin x - Bx\cos x + 2C\cos x - Cx\sin x,$$

$$y''' = 8Ae^{2x} - 3B\cos x + Bx\sin x - 3C\sin x - Cx\cos x,$$

and

$$\begin{aligned} y''' + 4y'' + y' + 4y &= [8A + 4(4A) + 2A + 4(A)]\,e^{2x} \\ &\quad + [-3B + 4(2C) + B + 4(0)]\cos x \\ &\quad + [-3C + 4(-2B) + C + 4(0)]\sin x \\ &\quad + [-C + 4(-B) + C + 4(B)]\,x\cos x \\ &\quad + [B + 4(-C) + (-B) + 4(C)]\,x\sin x \\ &= 30Ae^{2x} + (8C - 2B)\cos x + (-2C - 8B)\sin x \end{aligned}$$

Hence, if $y''' + 4y'' + y' + 4y = 5e^{2x} + 34\sin x$, then by equating coefficients

$$\begin{cases} 30A = 5 \\ 8C - 2B = 0 \\ -2C - 8B = 34 \end{cases} \implies \cdots \implies \begin{cases} A = 1/6 \\ B = -4 \\ C = -1. \end{cases}$$

Answer:

$$y = c_1e^{-4x} + c_2\cos x + c_3\sin x + \tfrac{1}{6}e^{2x} - 4x\cos x - x\sin x$$

in which c_1, c_2 and c_3 are arbitrary constants.

5. Substitute $y = xv$. Then $y' = xv' + v$, $y'' = xv'' + 2v'$, $y''' = xv''' + 3v''$, and

$$\begin{aligned} &x^3y''' - 3x^2y'' + x(6 - x^2)y' - (6 - x^2)y \\ &= x^3\left(xv''' + 3v''\right) - 3x^2\left(xv'' + 2v'\right) + x(6 - x^2)\left(xv' + v\right) - (6 - x^2)xv \\ &= x^4v''' + \left(3x^3 - 3x^3\right)v'' + \left[-6x^2 + x^2(6 - x^2)\right]v' + \left[x(6 - x^2) - (6 - x^2)x\right]v \\ &= x^4\left(v''' - v'\right). \end{aligned}$$

So the equation becomes

$$v''' - v' = 0.$$

This is a homogeneous linear equation with constant coefficients. Its auxiliary equation is

$$0 = m^3 - m = m\left(m^2 - 1\right) = m(m-1)(m+1).$$

Therefore its general solution is

$$v = c_1 + c_2 e^x + c_3 e^{-x}$$

where c_1, c_2 and c_3 are arbitrary constants.

Answer:

$$y = c_1 x + c_2 x e^x + c_3 x e^{-x}$$

in which c_1, c_2 and c_3 are arbitrary constants.

6. (a) By substitution,

$$x^2 y_1'' - x(x+2)y_1' + (x+2)y_1 = x^2(0) - x(x+2)1 + (x+2)x = 0$$

and

$$\begin{aligned} & x^2 y_2'' - x(x+2)y_2' + (x+2)y_2 \\ &= x^2(xe^x + 2e^x) - x(x+2)(xe^x + e^x) + (x+2)xe^x \\ &= x(x+2)e^x[x - (x+1) + 1] = 0. \end{aligned}$$

(b) The Wronskian of y_1 and y_2 is

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & xe^x \\ 1 & xe^x + e^x \end{vmatrix} = x(xe^x + e^x) - xe^x = x^2 e^x \neq 0 \quad \text{for } x \neq 0.$$

(c) By the method of variation of parameters the solution is given by

$$y = Ay_1 + By_2 = Ax + Bxe^x,$$

where

$$\begin{cases} A'y_1 + B'y_2 = A'x + B'xe^x = 0 \\ A'y_1' + B'y_2' = A' + B'(xe^x + e^x) = \dfrac{2x^3}{x^2} = 2x \end{cases}$$

$$\Longrightarrow \quad \cdots \quad \Longrightarrow$$

$$\begin{cases} A' = -2 \\ B' = 2e^{-x} \end{cases} \Longrightarrow \begin{cases} A = -2x + k_1 \\ B = -2e^{-x} + k_2 \end{cases}$$

where k_1 and k_2 are arbitrary constants. Hence, the general solution is

$$\begin{aligned} y &= \left(-2x + k_1\right)x + \left(-2e^{-x} + k_2\right)xe^x = -2x^2 + k_1 x - 2x + k_2 x e^x \\ &= \left(k_1 - 2\right)x + k_2 x e^x - 2x^2 = c_1 x + c_2 x e^x - 2x^2 \end{aligned}$$

in which c_1 and c_2 are arbitrary constants.