Dept. of Mathematics

Kuwait University

First In-Term Exam - Math 240
Ordinary Differential Equations

Spring Semester 2011/2012 **March 20, 2012**

Duration: 90 min.
Total marks: 25.
Calculators and ALL communication devices are NOT allowed during the examination.

Instructions: Answer all questions.

1. **(4 pts.) Find the particular solution of**

$$(x+y-2)\,dx-(x+4y-2)\,dy=0, \qquad \text{when } y=1 \text{ for } x=2.$$

2. **(4 pts.) Find the particular solution of**

$$(x-\sin y)\,dy+\tan y\;dx=0, \qquad \text{when } y(1)=\frac{\pi}{6}.$$

3. **(4 pts.) Solve the differential equation**

$$xy'-\frac{y}{\ln x}=xy^2.$$

4. **(4 pts.) Solve the differential equation**

$$\left(2e^{2y}-x\right)dx-2\left(e^{4y}+2xe^{2y}\right)dy=0.$$

5. **(4 pts.) Show the following ODE is Exact and solve it by the method of Exact ODE:**

$$\left(y^2+3x^2+e^{-y^2}\right)dx+2y\left(3y+x-xe^{-y^2}\right)dy=0.$$

6. **(2 pts.) If** $h(x,y)=\dfrac{5x}{\sqrt{x^2+3y^2}}$, **is it true that** h **is a homogeneous function? Justify your answer.**

7. **(3 pts.) If** $u=u(x,y)$ **is an Integrating Factor of the ODE** $M(x,y)\,dx+N(x,y)\,dy=0$, **show that:**

$$\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)u=N\frac{\partial u}{\partial x}-M\frac{\partial u}{\partial y}.$$

Solutions:

Q.1. Coeffs. linear in x & y: $\begin{cases} x+y-2=0 \\ x+4y-2=0 \end{cases} \Rightarrow \begin{cases} x=2 \\ y=0 \end{cases}$.

SUB: $u = x-2$, $du = dx$

$\Rightarrow (u+y)du - (u+4y)dy = 0$ (HOM. COEFFS. deg. 1)

$v = \frac{u}{y}$ (Note $y \not\equiv 0$ due to initial condition)

$\Rightarrow (v+1)y(v\,dy + y\,dv) - (v+4)y\,dy = 0$

$(v^2-4)dy + (v+1)y\,dv = 0$ (SEP)

$\frac{dy}{y} + \frac{v+1}{v^2-4}dv = 0$ (Note $v^2-4 \not\equiv 0$ due to initial cond.)

$\Rightarrow \frac{dy}{y} + \frac{1}{4}\left(\frac{3}{v-2} + \frac{1}{v+2}\right)dv = 0$

$4\ln|y| + 3\ln|v-2| + \ln|v+2| = \ln|C|$

$y^4(v-2)^3(v+2) = C \Rightarrow (x-2-2y)^3(x-2+2y) = C$

Initial condition: $\begin{cases} x=2 \\ y=1 \end{cases} \Rightarrow C = -16$. ※

Q.2. $M_y - N_x = \sec^2 y - 1 = \tan^2 y$

$\frac{1}{M}(M_y - N_x) = \tan y$

I.F. $u = e^{-\int \tan y\,dy} = e^{-\ln|\sec y|} = \cos y > 0$

$\Rightarrow \cos y\,(x - \sin y)dy + \sin y\,dx = 0$ (EXACT)

$d(x\sin y) - \frac{1}{2}\sin 2y\,dy = 0$

$x\sin y + \frac{1}{4}\cos 2y = C$

Initial condition: $\left.\begin{matrix} x=1 \\ y=\pi/6 \end{matrix}\right\} \Rightarrow C = \frac{1}{2} + \frac{1}{8} = \frac{5}{8}$. ※

Q.3. (BER) $n=2$, $z=y^{-1}$ (we exclude $y\equiv 0$)

$$dz = -y^{-2}dy$$

$$\Rightarrow dz + \frac{1}{x\ln x} z\,dx = -dx \quad (LIN)$$

I.F. $u = e^{\int \frac{dx}{x\ln x}} = e^{\ln|\ln x|} = \ln x > 0$

$$d(z\ln x) = -\ln x\,dx$$

$$z\ln x = -\int \ln x\,dx \overset{\text{by parts}}{=} x - x\ln x + c$$

$$\Rightarrow \ln x = y(x - x\ln x + c). \quad \#$$

Q.4. (SUB) $z = e^{2y}$, $dz = 2e^{2y}dy$

$$\Rightarrow (2z - x)dx - (z+2x)dz = 0 \quad (\text{HOM. c. deg. 1})$$

$$v = \frac{x}{z} \Rightarrow (2-v)z(v\,dz + z\,dv) - (1+2v)z\,dz = 0$$

$$-(v^2+1)dz + (2-v)z\,dv = 0$$

$$\Rightarrow \frac{dz}{z} + \frac{v-2}{v^2+1}dv = 0$$

$$\ln|z| + \frac{1}{2}\ln(v^2+1) - 2\tan^{-1}v = c$$

$$\Rightarrow \frac{1}{2}\ln(x^2+e^{4y}) - 2\tan^{-1}\left(\frac{x}{e^{2y}}\right) = c. \quad \#$$

Q.5. $M_y = 2y - 2y e^{-y^2} = N_x$

$$\begin{cases} ① \; F_x = M = y^2 + 3x^2 + e^{-y^2} \\ ② \; F_y = N = 6y^2 + 2yx - 2yxe^{-y^2} \end{cases}$$

$$① \Rightarrow F = y^2x + x^3 + xe^{-y^2} + K(y)$$

$$\Rightarrow F_y = 2yx - 2yxe^{-y^2} + K'(y) \overset{②}{=} 6y^2 + 2yx - 2yxe^{-y^2} \Rightarrow K'(y) = 6y^2$$

$$\Rightarrow K(y) = 2y^3 \Rightarrow F = \boxed{y^2x + x^3 + xe^{-y^2} + 2y^3 = C}$$

G.S. #

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Q.6. $$h(\lambda x, \lambda y) = \frac{5\lambda x}{\sqrt{\lambda^2 x^2 + 3\lambda^2 y^2}}$$
$$= \frac{5x}{\sqrt{x^2+3y^2}} \quad (\forall \lambda > 0)$$
$$= h(x,y)$$
$\Longrightarrow$ h is homogeneous of degree 0. (TRUE) #

Q.7. u is I.F. of $M dx + N dy = 0$

$\Rightarrow$ $uM dx + uN dy = 0$ is EXACT

$\Rightarrow$ $(uM)_y = (uN)_x$

$u_y M + M_y u = u_x N + N_x u$

$\Rightarrow$ $(M_y - N_x) u = N u_x - M u_y$. (QED) #
