KUWAIT UNIVERSITY
Department of Mathematics

Math 240 — Second Midterm — December 12th, 2011
Differential Equations — Time: 90 minutes

Use of calculators and/or mobile phones are not allowed in this exam.
Answer all the questions. Each question is worth 5 points

Problem 1. Determine all the solutions of

$$(D^2+4)(D^6-3D^4+3D^2-1)y=0.$$

Problem 2. Find the general solution of the equation

$$x^2y''-3xy'+4y=x$$

given that $y_1=x^2$ is a solution of the corresponding homogeneous equation.

Problem 3. Use the undetermined coefficients method to obtain a particular solution of

$$y^{(4)}+y'=x^2(e^{-x}+\sin(2x))$$

(P.S. **Do not determine the coefficients.**)

Problem 4. By using the substitution $u=e^{x+y}$, solve the initial value problem

$$y''+(y')^2+2y'+2=0, \quad y(0)=y'(0)=0.$$

Problem 5. Let a, b be real numbers, let x_0 be a fixed number. Consider

$$y''+ay'+by=0. \tag{1}$$

(i) Show that if $y(x_0)=y'(x_0)=0$, then $y=0$ is the only solution of (1).
(ii) Explain why in problem 4 we did not obtain the solution $y=0$.

1.

$m^2+4=0 \Rightarrow m = \mp 2i$

<u>**Soln**</u> : $0=(m^2-1)^3 \;\Rightarrow\; m=\pm1,\pm1,\pm1 \;\Rightarrow\; y=y_c=e^{-x}(b_0+b_1x+b_2x^2)+e^x(b_3+b_4x+b_5x^2)+C_1\cos 2x$ $+C_2\sin 2x$

2. **Find** the general solution of the differential equation $x^2y''-3x\,y'+4y=x$, given $y_1=x^2$

<u>**Soln**</u> : $y=v\,x^2 \Rightarrow \begin{cases} y'=v'\,x^2+2xv \\ y''=v''\,x^2+4xv'+2v \end{cases} \Rightarrow x^4v''+x^3v'=x \Rightarrow w'+\dfrac{w}{x}=\dfrac{1}{x^3}\;,\;\; \mu=e^{\int\frac{dx}{x}}=x$

$$w=\frac{1}{x}\left(\int\frac{dx}{x^2}+c\right)=\frac{1}{x}\left(-\frac{1}{x}+c\right)=-\frac{1}{x^2}+\frac{c}{x} \;\Rightarrow\; v=\frac{1}{x}+c\ln|x|+\beta \;\Rightarrow\; y=x+cx^2\ln|x|+\beta x^2$$

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3. Use the **undetermined coefficients** method to obtain the ***particular*** solution of $y^{(4)}+y'=x^2(e^{-x}+\sin 2x)$

Do not determine the coefficients. .

<u>**Soln**</u> : $0=m^4+m=m(m^3+1)=m(m+1)(m^2-m+1) \;\Rightarrow\; m=0,-1,(1\pm i\sqrt{3})/2$

$R(x)=x^2(e^{-x}+\sin 2x) \;\Rightarrow\; n=-1,-1,-1,\pm 2i,\pm 2i,\pm 2i$

$y_p=xe^{-x}(b_0+b_1x+b_2x^2)+\cos 2x\,(b_3+b_4x+b_5x^2)+\sin 2x\,(b_6+b_7x+b_8x^2)$

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4. Use sub $w=e^{x+y}$ to Solve IVP $\;y''+(y')^2+2y'+2=0,\;\; y'(0)=y(0)=0$

<u>**Soln**</u> : $w=e^{x+y} \;\Rightarrow\; \begin{cases} w(0)=e^0=1 \\ w'=(1+y')\,e^{x+y} \qquad \Rightarrow \qquad w'(0)=(1+0)e^0=1 \\ w''=y''e^{x+y}+(1+y')^2\,e^{x+y}=\left(y''+(y')^2+2y'+1\right)e^{x+y} \end{cases}$

$\Rightarrow$ *new IVP* $\;w''+w=0\quad, w'(0)=w(0)=1 \;\Rightarrow\; 0=m^2+1 \;\Rightarrow\; m=\pm i \;\Rightarrow\; w=w_c=c_1\sin x+c_2\cos x$

$\because\; 1=w(0) \;\Rightarrow\; c_2=1 \;\Rightarrow\; w=c_1\sin x+\cos x \;\Rightarrow\; w'=c_1\cos x-\sin x\;,$

$\because\; 1=w'(0) \;\Rightarrow\; c_1=1 \;\Rightarrow\; w=\sin x+\cos x=e^{x+y} \;\Rightarrow\; e^y=(\sin x+\cos x)e^{-x} \;\Rightarrow\; y=\ln(\sin x+\cos x)-x$

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5. **a)** For $\alpha,\beta\in\Re$ **Show that** $y=0$ is the only solution of IVP (*) $\;y''+\alpha\,y'+\beta\,y=0,\;\; y'(0)=y(0)=0$

$y=0$ is a solution of IVP (*) **[** $\because y=0 \;\Rightarrow\; \begin{cases} y'=y''=0 \;\Rightarrow\; 0+0+0=0 \\ y(0)=0=y'(0) \end{cases}$ **]**

But IVP (*) has unique solution ($\alpha,\beta\in\Re$, existence -uniqueness Th), $\therefore\; y=0$ is the only solution of IVP (*)

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b) explain why $y=0$ **is *not*** solution of IVP given in Q4

suppose $y=0$ is solution of IVP given in Q4 $\;\Rightarrow\; y'=y''=0 \;\Rightarrow\; 0+0^2+2(0)+2=2\neq 0$ contradiction