Dept. of Mathematics **Kuwait University**

Math 240
Ordinary Differential Equations

Fall Semester 2011/2012 **First In-Term Exam** **October 26, 2011**

Duration: 90 min.
Total marks: 25.
Calculators and ALL communication devices are NOT allowed during the examination.

Instructions: Answer all questions. Each question is worth 5 points.

Solve the initial value problem:

1. $y^2 dx + (x^2 + 3xy + 4y^2)\, dy = 0; \qquad x = 0,\ y = e^2.$

Solve the following differential equations:

2. $xdy - \left(y + 3xy^3 e^{x^3}\right) dx = 0.$

3. $\left[1 + (x+y)^2\right] dx + \left[1 + x\,(x+y)\right] dy = 0.$

4. $\left(2 - \dfrac{3}{x}\cos^2 y\right) dx + x^2 \tan y dy = 0.$

Answer the following question:

5. Given the following differential equation

$$y = x + y' - \ln y'$$

   (i) Is the equation linear? Give reason.

   (ii) Solve the differential equation by using the substitution $z = y'$.

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1. The equation is homogeneous. Let $x = vy \implies dx = vdy + ydv \implies$
$y^2(vdy + ydv) + (v^2y^2 + 3vy^2 + 4y^2)\,dy = 0 \implies ydv + (v^2 + 4v + 4)\,dy = 0$
$\frac{dy}{y} + \frac{dv}{(v+2)^2} = 0 \implies \ln|y| = (v+2)^{-1} + c \implies \boxed{\ln|y| = y(x+2y)^{-1} + c.}$ with $c = \frac{3}{2}$.

2. The equation is Bernoulli; $y' - \frac{1}{x}y = 3y^3e^{x^3}$. Let $z = y^{-2} \implies z' = -2y^{-3}y'$
$z' + \frac{2}{x}z = -6e^{x^3}$: Linear equation in z; $\mu(x) = \exp\left(\int \frac{2}{x}dx\right) = x^2$
$x^2z = \int -6x^2e^{x^3}dx + c = -2e^{x^3} + c \implies \boxed{x^2y^{-2} = -2e^{x^3} + c.}$

3. Let $z = x + y \implies dz = dx + dy \implies (1+z^2)\,dx + (1+xz)(dz - dx) = 0$
$(z^2 - xz)\,dx + (1+xz)\,dz = 0 \implies z^2dx + xzdz - xzdx + dz = 0$
$(zdx + xdz) - xdx + \frac{1}{z}dz = 0 \implies xz - \frac{1}{2}x^2 + \ln|z| = c$
$\boxed{x(x+y) - \frac{1}{2}x^2 + \ln|x+y| = c.}$

4. Multiply by $\sec^2 y \implies \left(2\sec^2 y - \frac{3}{x}\right)dx + x^2\sec^2 y\tan y dy = 0.$

Let $z = \sec^2 y \implies dz = 2\sec^2 y\tan y dy. \implies \left(4z - \frac{6}{x}\right)dx + x^2dz = 0 \implies$
$\frac{dz}{dx} + 4\frac{z}{x^2} = \frac{6}{x^3}$: Linear equation in z; $\mu(x) = \exp\left(\int \frac{4}{x^2}dx\right) = \exp(-\frac{4}{x})$
$\implies z\exp\left(-\frac{4}{x}\right) = \int \frac{6}{x^3}\exp\left(-\frac{4}{x}\right)dx + c = -\frac{3}{8}\exp\left(-\frac{4}{x}\right)\left[-\frac{4}{x} - 1\right] + c\,.$
$\boxed{\sec^2 y\exp\left(-\frac{4}{x}\right) = -\frac{3}{8}\exp\left(-\frac{4}{x}\right)\left[-\frac{4}{x} - 1\right] + c.}$

5. (ii) Using the substitiution $z = y' \implies z = 1 + z' - \frac{z'}{z}$; Is seperable $\implies$
$z' = \frac{(z-1)}{\left(1 - \frac{1}{z}\right)} = \frac{(z-1)}{\frac{1}{z}(z-1)} \implies \frac{dz}{z} = dx \implies \ln|z| = x + k \implies$
$y' = c_1e^x \implies \boxed{y = c_1e^x + c_2}$; where $c_1 = e^k, c_2$ are constants.