Dept. of Mathematics **Kuwait University**

MATH 240

Ordinary Differential Equations

Summer Semester 2011 **In-Term Test 1** **Tuesday, 05 July 2011**

Duration: 90 min.
Total marks: 25.
Calculators and ALL communication devices are NOT allowed during the examination.
Instructions: Answer the following questions. Remember to show all work in order to receive full credit.

1. [2 pts.] Given that $y = \Phi(x)$, prove (without solving the ODE) that $\Phi(x) = \sin x - \cos x$ is a solution to the initial value problem :

$$\frac{d^2y}{dx^2} + y = 0; \qquad y(0) = -1, \quad y'(0) = 1$$

2. [4 pts. each] Solve the first order differential equations:

a. $(1 + 4xy - 4x^2y)dx + (x^2 - x^3)dy = 0;$ when $x = 2,\ y = \frac{1}{4}.$

b. $[\sin(xy) + xy\cos(xy) + 2y]dx + [x^2\cos(xy) + 2x]dy = 0.$

c. $\dfrac{2x}{y^3}dx + \dfrac{(y^2 - 3x^2)}{y^4}dy = 0;$ when $y(1) = 1.$

d. $(2x - 3y + 1)dx + (5x - 4y + 13)dy = 0$

e. $(2xy^2 - 3y^3)dx + (7 - 3xy^2)dy = 0$

3. [3 pts.] Find the orthogonal trajectories of the given family of curves

$$y^2 = 2\ln(kxy), \qquad \text{where } k, x > 0,\ y > 1$$

passing through the point $P(1, 2)$, where k is a parameter.

Summer Semester 2011 In-Term Test 1 Key Solution

1. Solve by substitution.

2.

a. Linear in $y \Rightarrow$ since $(1+4xy(1-x))dx + x^2(1-x)dy = 0 \Rightarrow$
$x^2\dfrac{dy}{dx} + 4xy = \dfrac{1}{x-1}$, then dividing by x^2 we have: $\dfrac{dy}{dx} + \dfrac{4y}{x} = \dfrac{1}{x^2(x-1)}$ with I.F.$= x^4$. Solving
$\Rightarrow x^4 y = \int \dfrac{x^2}{(x-1)}dx \Rightarrow x^4 y = \ln|x-1| + \dfrac{x^2}{2} + x + c$ with $c = 0$. Then the solution is:
$\boxed{x^4 y = \ln|x-1| + x^2 + x}$.

b. Exact $\Rightarrow$ where $\dfrac{\delta M}{\delta y} = \dfrac{\delta N}{\delta x} = 2x\cos(xy) - x^2\sin(xy) + 2$. Let $F_y = N = x^2\cos(xy) + 2x$,
integrating w.r.t. $y \Rightarrow F(x,y) = x\sin(xy) + 2xy + h(x)$. Next, differentiating w.r.t.
$x \Rightarrow F_x(x,y) = \sin(xy) + xy\cos(xy) + 2y + h'(x)$, therefore, $h'(x) = 0 \Rightarrow h(x) = k$, then the solution is: $\boxed{F(x,y) = x\sin(xy) + 2xy + k}$

c. Bernoulli in $x \Rightarrow \dfrac{dx}{dy} + \dfrac{y^3}{2x}\left(\dfrac{(y^2-3x^2)}{y^4}\right) = 0 \Rightarrow \dfrac{dx}{dy} - \dfrac{3}{2}xy^{-1} = \dfrac{-1}{2}yx^{-1}$. Let $z = x^2$, then
$\dfrac{dz}{dy} = 2x\dfrac{dx}{dy}$. Subsituting in the D.E. $\Rightarrow \dfrac{dz}{dy} - 3zy^{-1} = -y$, with I.F.$= y^{-3}$. Solving
$\Rightarrow y^{-3}z = -\int y^{-2}dy \Rightarrow \boxed{y^{-3}z = \dfrac{1}{y} + k \text{ with } k = 0}$.

d. D.E. with linear coefficients in x & $y \Rightarrow$ For $l_1 = 2x - 3y + 1$ & $l_2 = 5x - 4y + 13$, l_1& l_2 intersect at $P(-5,-3)$. Choose change of variables: $x = s - 5, y = t - 3 \Rightarrow (2s-3t)ds + (5s-4t)dt = 0$.
Solve as homog. in s & $t \Rightarrow$
Let $z = \dfrac{s}{t} \Rightarrow \dfrac{1}{t}dt + \dfrac{2z-3}{2(z^2+z-2)}dz = 0$. Integrating,
$\ln|t| + \frac{7}{6}\ln|z+2| - \frac{1}{6}\ln|z-1| = c \Rightarrow \boxed{\ln|y+3| + \frac{7}{6}\ln\left|\dfrac{x+5}{y+3} + 2\right| - \frac{1}{6}\ln\left|\dfrac{x+5}{y+3} - 1\right| = c}$

e. I.F. by inspection $\Rightarrow -3y^2(ydx + xdy) + 2xy^2dx + 7dy = 0 \Rightarrow$

$(ydx + xdy) - \dfrac{2}{3}xdx - \dfrac{7}{3y^2}dy = 0$. Integrating, we have $\Rightarrow \boxed{xy - \dfrac{x^2}{3} + \dfrac{7}{3y} = k}$.

3. Differentiate the w.r.t. $x \Rightarrow \dfrac{dy}{dx} = \dfrac{y}{x(y^2-1)} \Rightarrow$ Then the orthogonal trajectories. D.E. is
$\dfrac{dx}{dy} = \dfrac{y}{x(1-y^2)}$. Solving, $\int xdx = \int \dfrac{y}{(1-y^2)}dy \Rightarrow$
$\dfrac{x^2}{2} = \dfrac{-1}{2}\int \dfrac{-2y}{(1-y^2)}dy$. The final solution is $\dfrac{x^2}{2} = -\frac{1}{2}\ln|1-y^2| + c$. Then the orthogonal trajectories to the equation that pass through $P(1,2)$ is $\boxed{\dfrac{x^2}{2} = -[y + \ln|y-1|] + \dfrac{1}{2}(\ln(3)+1)}$.