

Duration: 120 min.

Total marks: 40.

Instructions: Answer the following questions & show all work in order to receive full credit.

1. [5 pts. each] Find the general solution for the following differential equations:

(a)  $[2 \cos(2x + y) - x^2]dx + [\cos(2x + y) + e^y]dy = 0, \quad y(\pi) = 0.$

(b)  $[x^2 + (\tan^{-1}y)^2]dx + \frac{x^2}{1 + y^2}dy = 0, \quad y(1) = 0.$

2. [5 pts.] Let  $y_1 = x$  be one solution of Legendre's equation

$$(1 - x^2)y'' - 2xy' + 2y = 0, \quad -1 < x < 1$$

Find the second linearly independent solution  $y_2$ .

3. [5 pts.] Solve using Laplace transform method:  $y''(t) + 4y(t) = -e$ ;  $y(0) = 3$ , and use the constraint  $y\left(\frac{\pi}{4}\right) = 0$ , to find the value of  $y'(0) = b$ .

4. [5 pts.] Use the step function  $\alpha(t - c)$  to find the Laplace transform of  $H(t)$ , where

$$H(t) = \begin{cases} t^2 - 2, & 0 \leq t < 1 \\ 2 - t, & 1 \leq t < 3 \\ 7, & t > 3 \end{cases}$$

5. [3+4 pts.] If  $f(s) = L\{F(t)\}$  &  $a > 0$ . Use the definition of Laplace transform to:

(a) show that  $f(as) = \frac{1}{a}L\left\{F\left(\frac{t}{a}\right)\right\}$ .

(b) Using part (a), prove that if  $\phi(t) = \int_0^t \sin\left(\frac{t-\beta}{a}\right) \cos\left(\frac{\beta}{a}\right) d\beta$  then

$$L\{\phi(t)\} = \frac{a^3 s}{((as)^2 + 1)^2}.$$

6. [4+4 pts.]

(a) Find a formula for the coefficients of the power series solutions near the origin for the following homogeneous equation

$$(1 + x^2)y'' - 4xy' + 6y = 0, \quad y(0) = \frac{1}{6}, \quad y'(0) = 4.$$

(b) Show that the coefficients in the power series solutions near the origin of the following nonhomogeneous equation

$$(1 + x^2)y'' - 4xy' + 6y = e^{2x}$$

are recursively defined by the relations:

$$a_n = \frac{\left[ \frac{2^{n-2}}{(n-2)!} - (n-4)(n-5)a_{n-2} \right]}{n(n-1)}, \quad n \geq 2.$$

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