Kuwait University | Department of Mathematics

Math 240 : Ordinary Differential Equations

Spring Semester 2011 | In-Term Test 2 | Wednesday 11 May 2011

Duration: 90 minutes.
Calculators and mobile telephones are NOT allowed.
Read each question carefully.
Show all your work in order to receive full credit.
Total marks: 25.

1. Let $A = (\cos x)D$ and $B = (\sin x)D$ where D denotes differentiation with respect to x. Find the differential operator $AB - BA$. [4 pts.]

2. Solve

   $$(D^4 - 8D^2 + 16)(3D^2 + 2D + 1)y = 0.$$ [4 pts.]

3. Find the general solution of

   $$(D^3 + 5D^2 + 3D - 9)y = e^{-3x}x^2 + 4x.$$ [4 pts.]

4. Solve

   $$(D - 2)(D^2 - 2D + 5)y = e^{2x}\sin 2x.$$ [4 pts.]

5. Verify that $y = e^x$ is a solution of the homogeneous equation associated with

   $$xy'' - (x + 1)y' + y = x^2e^{2x} \quad \text{for } x \neq 0,$$

   and find the general solution. [4 pts.]

6. Find the general solution of

   $$y'' + 4y = 3\csc x.$$ [5 pts.]

Math 240 : Ordinary Differential Equations

Spring Semester 2011 SOLUTIONS In-Term Test 2

1.
$$\begin{aligned}(AB-BA)y &= \cos x\frac{d}{dx}\left(\sin x\frac{dy}{dx}\right)-\sin x\frac{d}{dx}\left(\cos x\frac{dy}{dx}\right)\\ &= \cos x\left(\cos x\frac{dy}{dx}+\sin x\frac{d^2y}{dx^2}\right)-\sin x\left(-\sin x\frac{dy}{dx}+\cos x\frac{d^2y}{dx^2}\right)\\ &= \cdots = \frac{dy}{dx}.\end{aligned}$$

Answer: $AB-BA=D$.

2. The auxiliary equation is

$$0=(m^4-8m^2+16)(3m^2+2m+1)=\cdots=3(m-2)^2(m+2)^2\left[\left(m+\tfrac{1}{3}\right)^2+\left(\tfrac{\sqrt{2}}{3}\right)^2\right].$$

This has roots $m=\pm 2$ with multiplicity 2, and $m=-\frac{1}{3}\pm\frac{\sqrt{2}}{3}i$.
Thus, the general solution of the differential equation is

$$y=c_1e^{2x}+c_2xe^{2x}+c_3e^{-2x}+c_4xe^{-2x}+c_5e^{-x/3}\cos\big(\sqrt{2}x/3\big)+c_6e^{-x/3}\sin\big(\sqrt{2}x/3\big)$$

where c_1, c_2, c_3, c_4, c_5, and c_6 are arbitrary constants.

3. The auxiliary equation is

$$0=m^3+5m^2+3m-9=\cdots=(m-1)(m+3)^2.$$

This has roots $m=1$, and $m=-3$ with multiplicity 2. Therefore, the complementary function is

$$y_c=c_1e^x+c_2e^{-3x}+c_3xe^{-3x}.$$

Using the method of inverse operators,* a particular solution of the inhomogeneous equation is given by

$$y_p+4y_q \quad\text{where}\quad y_p\doteq\frac{1}{(D-1)(D+3)^2}e^{-3x}x^2 \quad\text{and}\quad y_q\doteq\frac{1}{(D-1)(D+3)^2}x.$$

Using exponential shift,

$$y_p\doteq e^{-3x}\frac{1}{(D-4)D^2}x^2.$$

*The symbol $\doteq$ is used to denote the formal computation used in the method.

Hence,

$$\begin{aligned}
y_p &\doteq -\frac{1}{4}e^{-3x}\frac{1}{D^2}\frac{1}{1-\dfrac{D}{4}}x^2 \doteq -\frac{1}{4}e^{-3x}\frac{1}{D^2}\left[1+\frac{D}{4}+\left(\frac{D}{4}\right)^2+\left(\frac{D}{4}\right)^3+\cdots\right]x^2 \\
&\doteq -\frac{1}{4}e^{-3x}\frac{1}{D^2}\left(1+\frac{1}{4}D+\frac{1}{16}D^2\right)x^2 = -\frac{1}{4}e^{-3x}\frac{1}{D^2}\left(x^2+\frac{1}{4}2x+\frac{1}{16}2\right) \\
&= -\frac{1}{4}e^{-3x}\frac{1}{D^2}\left(x^2+\frac{x}{2}+\frac{1}{8}\right) \doteq -\tfrac{1}{4}e^{-3x}\frac{1}{D}\left(\frac{x^3}{3}+\frac{x^2}{4}+\frac{x}{8}\right) \\
&\doteq -\frac{1}{4}e^{-3x}\left(\frac{x^4}{12}+\frac{x^3}{12}+\frac{x^2}{16}\right) = -\frac{1}{192}x^2(4x^2+4x+3)e^{-3x}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
y_q &\doteq \frac{1}{D^3+5D^2+3D-9}x \doteq -\frac{1}{9}\frac{1}{1-\dfrac{3D+5D^2+D^3}{9}}x \\
&\doteq -\frac{1}{9}\left[1+\frac{3D+5D^2+D^3}{9}+\left(\frac{3D+5D^2+D^3}{9}\right)^2+\cdots\right]x \\
&\doteq -\frac{1}{9}\left(1+\frac{1}{3}D\right)x = -\frac{1}{9}\left(x+\frac{1}{3}\right) = -\frac{1}{27}(3x+1).
\end{aligned}$$

Answer: The general solution of the inhomogeneous equation is

$$y = c_1e^x + c_2e^{-3x} + c_3xe^{-3x} - \tfrac{1}{192}(4x^2+4x+3)x^2e^{-3x} - \tfrac{4}{27}(3x+1)$$

where c_1, c_2, and c_3 are arbitrary constants.

4. The auxiliary equation is

$$0 = (m-2)(m^2-2m+5) = (m-2)\left[(m-1)^2+2^2\right].$$

This has roots $m = 2$ and $m = 1 \pm 2i$. Therefore, the complementary function is

$$y_c = c_1e^{2x} + c_2e^x\cos 2x + c_3e^x\sin 2x.$$

Using the method of inverse operators with exponential shift at the first step, a particular solution of the inhomogeneous equation is given by

$$
\begin{aligned}
y_p &\doteq \frac{1}{(D-2)\left[(D-1)^2+2^2\right]} e^{2x}\sin 2x \doteq e^{2x}\frac{1}{D\left[(D+1)^2+2^2\right]}\sin 2x \\
&\doteq e^{2x}\frac{1}{D^3+2D^2+5D}\sin 2x \doteq e^{2x}\frac{1}{(-2^2)D+2(-2^2)+5D}\sin 2x \\
&\doteq e^{2x}\frac{1}{D-8}\sin 2x \doteq e^{2x}(D+8)\frac{1}{D^2-8^2}\sin 2x \doteq e^{2x}(D+8)\frac{1}{-2^2-8^2}\sin 2x \\
&= -\frac{1}{68}e^{2x}(D+8)\sin 2x = -\frac{1}{68}e^{2x}(2\cos 2x+8\sin 2x) \\
&= -\frac{1}{34}e^{2x}(\cos 2x+4\sin 2x).
\end{aligned}
$$

Answer: The general solution of the inhomogeneous equation is

$$y = c_1e^{2x} + c_2e^{x}\cos 2x + c_3e^{x}\sin 2x - \tfrac{1}{34}e^{2x}(\cos 2x + 4\sin 2x)$$

where c_1, c_2, and c_3 are arbitrary constants.

5. Use the method of reduction of order. Set

$$y = e^x v.$$

Then

$$y' = e^x v + e^x v', \quad y'' = e^x v + 2e^x v' + e^x v'',$$

and substitution in the equation gives

$$
\begin{aligned}
x^2e^{2x} &= x(e^x v + 2e^x v' + e^x v'') - (x+1)(e^x v + e^x v') + e^x v = \cdots \\
&= xe^x v'' + (x-1)e^x v'.
\end{aligned}
$$

The cancellation of the terms involving v verifies that e^x is a solution of the homogeneous equation.
The above equation simplifies to

$$v'' + \frac{x-1}{x}v' = xe^x.$$

This is linear. An integrating factor can be found from

$$e^{\int \frac{x-1}{x}\,dx} = e^{\int 1-\frac{1}{x}\,dx} \doteq e^{x-\ln|x|} = e^x/|x|.$$

Multiplying by e^x/x, the equation becomes

$$\frac{d}{dx}\left(\frac{e^x}{x}v'\right) = e^{2x}.$$

Hence,

$$\frac{e^x}{x}v' = \tfrac{1}{2}e^{2x} + c_1, \quad v' = \tfrac{1}{2}xe^x + c_1 xe^{-x}, \quad \text{and} \quad v = \tfrac{1}{2}(x-1)e^x - c_1(x+1)e^{-x} + c_2.$$

Replacing c_1 with $-c_1$, this gives the general solution of the inhomogeneous equation

$$y = c_1(x+1) + c_2 e^x + \tfrac{1}{2}(x-1)e^{2x}$$

where c_1 and c_2 are arbitrary constants.

6. The auxiliary equation is $m^2 + 4 = 0$ which has roots $m = \pm 2i$ and leads to the complementary function

$$y_c = c_1 \cos 2x + c_2 \sin 2x.$$

According to the method of variation of parameters, the solution of the inhomogeneous equation is given by

$$y = A\cos 2x + B \sin 2x,$$

where

$$A' \cos 2x + B' \sin 2x = 0, \tag{1}$$

and

$$A' \frac{d}{dx} \cos 2x + B' \frac{d}{dx} \sin 2x = 3 \csc x,$$

which simplifies to

$$-2A' \sin 2x + 2B' \cos 2x = 3 \csc x. \tag{2}$$

Multiplying (1) by $\cos 2x$ and (2) by $\frac{1}{2} \sin 2x$ and subtracting gives

$$A' = -\tfrac{3}{2} \sin 2x \csc x = \cdots = -3 \cos x.$$

Multiplying (1) by $\sin 2x$ and (2) by $\frac{1}{2} \cos 2x$ and adding gives

$$B' = \tfrac{3}{2} \cos 2x \csc x = \cdots = \tfrac{3}{2} \csc x - 3 \sin x.$$

Hence,

$$A = -3 \sin x + c_1 \quad \text{and} \quad B = \pm\tfrac{3}{2} \ln |\csc x \mp \cot x| + 3 \cos x + c_2.$$

This yields the general solution of the inhomogeneous equation

$$\begin{aligned} y &= \left(-3 \sin x + c_1\right) \cos 2x + \left(\pm\tfrac{3}{2} \ln |\csc x \mp \cot x| + 3 \cos x + c_2\right) \sin 2x = \cdots \\ &= c_1 \cos 2x + c_2 \sin 2x + 3 \sin x \left(1 \pm \cos x \ln |\csc x \mp \cot x|\right) \end{aligned}$$

where c_1 and c_2 are arbitrary constants.