Kuwait University	Math 240	April 11, 2011
Dept. of Math.	First Midterm	Duration: 90 minutes

Calculators and mobile phones are not allowed.
Answer all of the following questions.

1. (4 pts) Solve the following differential equation:

$$(xy - y^2)dx + (x^2 - xy - x + y)dy = 0.$$

2. (4 pts) Solve the following differential equation:

$$y' = \frac{1}{e^{-y} - x}.$$

3. (4 pts) Solve the initial value problem:

$$(y^3 - xy)dx + 2y^2 dy = 0, \qquad y(0) = 1.$$

4. (4 pts) Solve the following differential equation:

$$\left(\frac{2}{y} + \frac{y}{x}\right)dx + \left(\frac{x}{y^2} + 3\right)dy = 0.$$

5. (4 pts) Find the orthogonal trajectories of the family of curves:

$$x^2 + y^2 = cy.$$

6. (5 pts) Solve the following differential equation:

$$\left(\frac{\sin(2x)}{2} + y\cos x + 2\cos x\right)dx + (\sin x - y)dy = 0.$$

Solution

1. $y(x-y)\,dx + (x-1)(x-y)\,dy = 0 \rightarrow -(y-x)[y\,dx + (x-1)\,dy] = 0$.
Hence, $y = x$, or, $y\,dx + (x-1)\,dy = 0$. The latter can be written as $d[y(x-1)] = 0$.
Therefore the general solution is $y = x$ or $y(x-1) = C$

2. $\dfrac{dy}{dx} = \dfrac{1}{e^{-y} - x}$ can be written as $\dfrac{dx}{dy} + x = e^{-y}$ which is linear with integrating factor $\mu(y) = e^y$. After multipilying by μ DE becomes: $e^y \dfrac{dx}{dy} + e^y x = 1$ which gives $\dfrac{d}{dy}(e^y x) = 1$. So $x = (y + C)e^{-y}$.

3. This is a Bernoulli equation which takes the form: $2yy' + y^2 = x$. Let $z = y^2$. DE becomes: z'+z=x which is linear with integrating factor $\mu = e^x$. So $(e^x z)' = xe^x$.
Integrating gives $z = x - 1 + Ce^{-x}$ which implies $y^2 = x - 1 + Ce^{-x}$.
Substituting using $y(0) = 1$ leads to $y = \sqrt{x - 1 + 2e^{-x}}$.
($y = -\sqrt{x - 1 + 2e^{-x}}$ is inadmissible since $y(0) = 1$)

4. DE is equivalent to $(2xy + y^3)\,dx + (x^2 + 3xy^2)\,dy = 0$ which is exact since $\dfrac{\partial M}{\partial y} = 2x + 3y^2 = \dfrac{\partial N}{\partial x}$. We seek a function $f(x, y)$ satisfying: $\dfrac{\partial f}{\partial x} = 2xy + y^3$ and $\dfrac{\partial f}{\partial y} = x^2 + 3xy^2$. Integrating the first equation gives: $f(x, y) = x^2 y + xy^3 + C(y)$.
Differentiating with respect to y and comparing with the second equation yields $C(y) = 0$.
Therefore the general solution of the DE is: $x^2 y + xy^3 = K$.

5. $d(\dfrac{x^2 + y^2}{y}) = 0$ gives $2xy\,dx - (x^2 - y^2)\,dy = 0$. To find the orthogonal trajectories we solve: $(x^2 - y^2)\,dx + 2xy\,dy = 0$ which is homogeneous. Let $z = \dfrac{y}{x}$
we get: $(1 - z^2)\,dx + 2z(xdz + zdx) = 0$ this separates to give: $-\dfrac{dx}{x} = \dfrac{2zdz}{z^2 + 1}$.
Integrating leads to $x^2 + y^2 = Kx$.

6. $(\sin x + y + 2)\cos x\,dx + (\sin x - y)\,dy = 0$. Let $t = \sin x$, $dt = \cos x\,dx$.
DE becomes: $(t + y + 2)\,dt + (t - y)\,dy = 0$. Let $t = u + h$ and $y = v + k$, then h and k satisfy
the system with equations $h - k + 2 = 0$ and $h - k = 0$. This gives $h = k = -1$. So $t = u - 1$
and $y = v - 1$. When substituting DE becomes: $(u + v)\,du + (u - v)\,dv = 0$ which is homogeneous.
Let $z = \dfrac{v}{u}$. DE becomes: $(1 + z)\,du + (1 - z)(u\,dz + z\,du) = 0$. This is separable of the form:

$-\frac{du}{u} = \frac{z-1}{z^2-2z-1}dz$ integrating leads to $-2\ln u + C = \ln(z^2-2z-1)$. By backtracking the
previous substitutions we finally obtain: $(y+1)^2 - 2(\sin x+1)(y+1) - (\sin x+1)^2 = K$.
Another way to solve question 6 is to notice that it is actually exact.
Namely, $d[\frac{1}{2}\sin^2 x + y\sin x + 2\sin x - \frac{1}{2}y^2] = 0$.
So, $\sin^2 x + 2y\sin x + 4\sin x - y^2] = K$ is the general solution.