Kuwait University	**Math 240**	**Date: Dec. 19th, 2010**
Department of Mathematics	**Second Exam**	**Duration: 90 min.**

ONLY pens, pencils, erasers and rulers are allowed.
You must show your work and all important steps to receive credit.

1. [**3 pts**] Determine if the two functions $f(x) = 2\cos x + 3\sin x$ and $g(x) = 3\cos x - 2\sin x$ are linearly independent in the interval $(0,1)$.

2. [**3 pts**] Let $A = (xD+2)$ and $B = \left(x^2D+1\right)$. Does $AB = BA$?

3. [**4 pts**] Find the general solution of the differential equation

$$(D+1)^2(4D^3+9D)y = 0.$$

4. [**5 pts**] Solve the initial value problem

$$x^2y'' - 3xy' + 4y = x^2 \ln x \; ; \qquad y(1) = 1,\; y'(1) = 0,$$

given that $h(x) = x^2$ is a solution of the corresponding homogeneous equation.

5. [**5 pts**] Use the method of variation of parameters to find the general solution of the differential equation

$$4\,y'' + y = 2\sec\left(\frac{x}{2}\right),$$

where $x \in (-\pi, \pi)$.

6. [**5 pts**] Use inverse differential operators to find a particular solution for the differential equation

$$(D^2+D-2)y = e^x\,(1+\cos x) - 3x.$$

Some useful formulas for inverse differential operators

$\frac{1}{f(D)}\left(e^{ax}u(x)\right)$	$=$	$e^{ax}\frac{1}{f(D+a)}u(x)$	$\frac{1}{g(D)(D-a)^k}e^{ax}$	$=$	$\frac{x^ke^{ax}}{k!g(a)}$,	$g(a) \neq 0$
$\frac{1}{D^2+b^2}\cos(bx)$	$=$	$\frac{x\sin(bx)}{2b}$	$\frac{1}{D^2+b^2}\sin(bx)$	$=$	$\frac{-x\cos(bx)}{2b}$	

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1. $W(f,g)(x) \neq 0$. Hence f and g are Linearly Independent

2. $ABy = A(By) = (xD+2)\ \left[x^2Dy + y\right] = x^3D^2y + 2x^2Dy + xDy + 2x^2Dy + 2y$
$= \boxed{x^3D^2y + (4x^2+x)Dy + 2y}$, and $BAy = B(Ay) = (x^2D+1)(xDy+2y) = x^3D^2y + x^2Dy + 2x^2Dy + xDy + 2y$
$= \boxed{x^3D^2y + (3x^2+x)Dy + 2y}$. Therefore $AB \neq BA$ $(ABx \neq BAx)$.

3. Auxiliary equation is $m(m+1)^2\left(4m^2+9\right) = 0$. The general solution is:

$$\boxed{y = c_1 + c_2e^{-x} + c_3xe^{-x} + c_4\sin\left(\tfrac{3x}{2}\right) + c_5\cos\left(\tfrac{3x}{2}\right)}.$$

4. Let $y = vx^2 \to y' = v'x^2 + 2xv \to y'' = v''x^2 + 4xv' + 2v$. Substitute into the differential equ. $\to \boxed{xv'' + v' = \dfrac{\ln x}{x}}$. Let $w = v' \to xw' + w = \dfrac{\ln x}{x} \to (xw)' = \dfrac{\ln x}{x} \to v' = w = \dfrac{(\ln x)^2}{2x} + \dfrac{c_1}{x} \to$
$v = \dfrac{(\ln x)^3}{6} + c_1\ln x + c_2 \to \boxed{y = c_1x^2\ln x + c_2x^2 + \dfrac{x^2(\ln x)^3}{6}}$. Using I.C., we get $c_1 = -2$ and $c_2 = 1$.
Hence $\boxed{y = -2x^2\ln x + x^2 + \dfrac{x^2(\ln x)^3}{6}}$.

5. Auxiliary equation is $4m^2 + 1 = 0 \to y_c = c_1\cos\left(\dfrac{x}{2}\right) + c_2\sin\left(\dfrac{x}{2}\right)$. Let $y_p = A(x)\cos\left(\dfrac{x}{2}\right) + B(x)\sin\left(\dfrac{x}{2}\right)$, where

$$A'\cos\left(\frac{x}{2}\right) + B'\sin\left(\frac{x}{2}\right) = 0.$$

Substituting in D.E.$\to$ $$-\tfrac{1}{2}A'\sin\tfrac{1}{2}x + \tfrac{1}{2}B'\cos\tfrac{1}{2}x = \tfrac{1}{2}\sec\left(\frac{x}{2}\right).$$

Solving we get: $A = 2\ln\left|\cos\left(\dfrac{x}{2}\right)\right|$ and $B = x$. Therefore the general solution is

$$y = c_1\cos\left(\frac{x}{2}\right) + c_2\sin\left(\frac{x}{2}\right) + 2\ln\left|\cos\left(\frac{x}{2}\right)\right|\cos\left(\frac{x}{2}\right) + x\sin\left(\frac{x}{2}\right).$$

6.

$$\begin{aligned}
y_p &= \frac{1}{(D+2)(D-1)}e^x + \frac{1}{(D+2)(D-1)}e^x\cos x + \frac{1}{-2+D+D^2}(-3x) \\
&= \frac{x}{1!(1+2)}e^x + \operatorname{Re}\left(\frac{1}{(D+2)(D-1)}e^{(1+i)x}\right) + (-\frac{1}{2} - \frac{D}{4})(-3x) \\
&= \frac{xe^x}{3} + \operatorname{Re}\left(\frac{1}{(1+i+2)(1+i-1)}e^{(1+i)x}\right) + \frac{3x}{2} + \frac{3}{4} = \frac{xe^x}{3} + e^x\operatorname{Re}\left(\frac{1}{-1+3i}e^{ix}\right) + \frac{3x}{2} + \frac{3}{4} \\
&= \frac{xe^x}{3} + e^x\operatorname{Re}\left(\frac{-1-3i}{10}e^{ix}\right) + \frac{3x}{2} + \frac{3}{4} = \frac{xe^x}{3} + \frac{e^x}{10}(-\cos x + 3\sin x) + \frac{3x}{2} + \frac{3}{4}.
\end{aligned}$$