KUWAIT UNIVERSITY
DEPARTMENT OF MATHEMATICS

Math 240 | MidTerm 1 | 4 November 2010
Ordinary Diff. Equations | | Time: 90 minutes

Calculators and cellular phones are not allowed during the exam.

1. Solve the following differential equations: [4 pts each]

   (a) $y' = (x+y)^2 + 2x + 2y$

   (b) $y' = y + e^x y^2$, where $y(0) = -1$

   (c) $2y\,dx + (x - \sin\sqrt{y})\,dy = 0$

   (d) $(1+t^2)\,du - (1 + 2tu + t^2)\,dt = 0$ where $u(1) = 2\pi$

   (e) $(x - \cos y)\,dx - (y - x\sin y)\,dy = 0$

2. Let F be a family of curves defined by the equation

$$x^2 + xy + y^2 = C$$

   Find the family G of its orthogonal trajectories. [5 pts]

1. (a) $y' = x^2 + 2xy + y^2 + 2x + 2y = (x+y)^2 + 2(x+y)$. Put $v = x + y$ and differentiate w.r.t x. Then $y' = v' - 1$ and we have the separable equation

$$v' = 1 + v^2 + 2v = (v+1)^2 \quad \longrightarrow \quad \frac{dv}{(v+1)^2} = dx \quad \longrightarrow \quad -\frac{1}{v+1} = x - C \quad \longrightarrow \quad \boxed{x + \frac{1}{x+y+1} = C}$$

(b) This is Bernoulli's equation

$$y' = y + e^x y^2 \quad \longrightarrow \quad y' - y = e^x y^2 \quad \longrightarrow \quad -y^{-2}y' + y^{-1} = -e^x$$

If $z = y^{-1}$, the equation becomes a linear equation in standard form $z' + z = -e^x$ with integrating factor e^x so that the general solution is given by

$$e^x z = \int_x -e^{2t}\,dt = -\frac{1}{2}e^{2x} + C \quad \longrightarrow \quad \frac{1}{y} = -\frac{1}{2}e^x + C\,e^{-x}$$

The initial condition $y(0) = -1$ gives $C = -1/2$ and the particular solution is $\boxed{y = -\operatorname{sech} x}$.

(c) The equation $2y\,dx + (x - \sin\sqrt{y})\,dy = 0$ is not exact since $M_y = 2$ and $N_x = 1$. But $M_y - N_x = 1$ and when this is divided by M it yields a function of y alone, namely, $1/(2y)$. We find the integrating factor to be

$$\rho(y) = \exp\left(-\int_y \frac{1}{2s}\,ds\right) = \frac{1}{\sqrt{y}}$$

Multiply the equation by $\rho(y)$ to make it exact:

$$dF = F_x\,dx + F_y\,dy = 2\sqrt{y}\,dx + \frac{x - \sin\sqrt{y}}{\sqrt{y}}\,dy = 0 \quad \longrightarrow \quad F_x = 2\sqrt{y} \quad \longrightarrow \quad F = 2x\sqrt{y} + g(y)$$

$$F_y = \frac{x}{\sqrt{y}} + g'(y) = \frac{x - \sin\sqrt{y}}{\sqrt{y}} \quad \longrightarrow \quad g'(y) = -\frac{\sin\sqrt{y}}{\sqrt{y}} \quad \longrightarrow \quad g(y) = 2\cos\sqrt{y}$$

and we have the family of solutions $\boxed{x\sqrt{y} + \cos\sqrt{y} = C}$.

(d) The equation is expressed as

$$u' - \frac{2t}{1+t^2}\,u = 1$$

for which the integrating factor is found to be

$$\rho = \exp\left[-\int_t \frac{2z}{1+z^2}\,dz\right] = \frac{1}{1+t^2} \quad \longrightarrow$$

$$\frac{1}{1+t^2}\,y = \int_t \frac{1}{1+z^2}\,dz = \tan^{-1} t + C \quad \longrightarrow \quad \boxed{u = (1+t^2)\,(\tan^{-1} t + C)}$$

where $\boxed{C = 3\pi/4}$ is obtained from the initial condition $u(1) = 2\pi$.

(e) The equation is exact and we have the general solution $\boxed{x^2 - y^2 - 2x\cos y = C}$

2. Take the derivative w.r.t. x

$$2x + y + x\,y' + 2y\,y' = 0 \quad \longrightarrow \quad y'_F = -\frac{2x+y}{x+2y} \quad \longrightarrow \quad y'_G = \frac{x+2y}{2x+y}$$

Hence, for the family G we have the following DE with homogeneous coefficients:

$$(x+2y)\,dx - (2x+y)\,dy = 0$$

Make the substitution $y = xv$ to transform the above into

$$(x+2xv)\,dx - (2x+xv)\,(vdx+xdv) = 0 \quad \longrightarrow \quad (1+2v)\,dx - (2+v)\,(vdx+xdv) = 0 \quad \longrightarrow$$

$$(1-v^2)\,dx - x\,(2+v)\,dv = 0 \quad \longrightarrow \quad \frac{dx}{x} + \frac{2+v}{v^2-1}\,dv = 0 \quad \longrightarrow \quad 2\,\frac{dx}{x} + \left[\frac{3}{v-1} - \frac{1}{v+1}\right] dv = 0 \quad \longrightarrow$$

$$\ln x^2 + \ln\left|\frac{(v-1)^3}{v+1}\right| = \ln|K| \quad \longrightarrow \quad x^2\,\frac{(v-1)^3}{v+1} = K \quad \longrightarrow \quad \boxed{(y-x)^3 = K\,(y+x)}$$