Kuwait University	Math 240	January 15, 2011
Department of Mathematics	Final Examination	Duration : 2 hours

The use of calculators of all kinds is not allowed. All communication devices including cell phones are prohibited. Answer all of the following questions. Read each question carefully.

1. Verify that the one-parameter family of curves $x^3 + y^3 = 3cxy$ are solutions of the differential equation
$$\frac{dy}{dx} = -\frac{y\left(2x^3 - y^3\right)}{x\left(2y^3 - x^3\right)}$$ **(2 points)**

2. Find all differentiable functions $f(x)$ on some interval I such that $W(x, xf(x)) = 1 + 2f(x)$. Here, W is the Wronskian of the functions x and $xf(x)$. **(3 points)**

3. Solve the initial value problem
$$x\frac{dy}{dx} + 2\left(\ln y - 2x^2\right)y = 0, \qquad y(1) = e^2$$ **(3 points)**

4. A certain homogeneous second-order linear differential equation has e^x and e^{-x} as solutions over $(-\infty, \infty)$. Find the value of the solution satisfying $y(0) = 0$, $y'(0) = 4$ at $x = 2$. **(3 points)**

5. Use the inverse differential operator method to find a particular solution to the differential equation
$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = xe^{-3x} + e^{-3x}\cos x$$ **(3 points)**

6. Consider the differential equation
$$x^2y'' + 3xy' - 3y = \frac{12x^2}{1 + x^2} \quad \text{for } x > 0$$
   (a) Find two linearly independent solutions of the form x^r to the associated homogeneous differential equation. **(3 points)**

   (b) Find the general solution of the given equation. **(3 points)**

7. Find $L\left\{te^{-2t}\sin 3t\right\}$. **(5 points)**

8. Solve the initial value problem
$$y'' + 4y = g(x) = \begin{cases} 0, & x < \frac{\pi}{2} \\ \sin x, & x \geq \frac{\pi}{2} \end{cases} \qquad y(0) = 1, \qquad y'(0) = 2$$ **(5 points)**

9. Solve the integral equation for $y(t)$
$$y(t) = 1 + \int_0^t y(\beta)\cos(t - \beta)d\beta$$ **(5 points)**

10. Find a power series solution of the initial value problem about the ordinary point $x = 0$ and specify its interval of validity
$$(x^2 - 1)y'' + 4xy' + 2y = 0, \quad y(0) = 4, \quad y'(0) = 1$$ **(5 points)**

Final Examination (January 15, 2011; Fall 10/11) Solution Key

1. Write the given family as $\frac{x^2}{y} + \frac{y^2}{x} = 3c$. Eliminate the parameter by differentiating with respect to x implicitly and obtain

$$\tfrac{2x}{y} - \tfrac{x^2}{y^2}y' + \tfrac{2yy'}{x} - \tfrac{y^2}{x^2} = 0, \qquad \text{which gives } \left(\tfrac{2y^3-x^3}{y^2x}\right)y' = \tfrac{y^3-2x^3}{x^2y}. \text{ Hence, } y' = -\tfrac{y\left(2x^3-y^3\right)}{x\left(2y^3-x^3\right)}$$

2. We have

$$W(x, xf(x)) = \begin{vmatrix} x & xf \\ 1 & xf'+f \end{vmatrix} = x^2f' + xf - xf = x^2f' = 1+2f$$

Solving the separable differential equation gives $\ln c\,|1+2f| = -\frac{2}{x}$, where $c>0$ and a constant solution $f = -\frac{1}{2}$. Hence, f is any of the functions

$$\begin{cases} f(x) = Be^{-\frac{2}{x}} - \frac{1}{2}, & x \in I \subset (0,\infty) \\ f(x) = Be^{-\frac{2}{x}} - \frac{1}{2}, & x \in I \subset (-\infty,0) \\ f(x) = -\frac{1}{2} & -\infty < x < \infty \end{cases} \qquad \text{where } B \geq 0.$$

3. Use the substitution $u = \ln y$ or $y = e^u$ to write the equation in terms of the dependent variable u as

$$\tfrac{du}{dx} + \tfrac{2}{x}u = 4x$$

The resulting equatuion is linear, whose integrating factor is $e^{\int \frac{2}{x}dx} = x^2$, and its general solution is $u = x^2 + cx^{-2}$. Hence, $y = e^{\left(x^2+cx^{-2}\right)}$. The initial value problem has the particular solution $y = e^{\left(x^2+x^{-2}\right)}$.

4. The solutions e^x and e^{-x} are linearly independent on $(-\infty,\infty)$. The required solution must have the form $y = Ae^x + Be^{-x}$. Hence, $y(0) = 0 = A+B$, and $y'(0) = 4 = A-B$. Solving for A and B gives $A = 2$ and $B = -2$. The solution is $y = 2e^x - 2e^{-x}$ and $y(2) = 2\left(e^2 - e^{-2}\right)$.

5. Writing $D = \frac{d}{dx}$ we have

$$\begin{aligned} y_p &= \tfrac{1}{(D+4)(D+3)}(x+\cos x)\,e^{-3x} = e^{-3x}\tfrac{1}{(D+1)D}(x+\cos x) = \tfrac{1}{2}e^{-3x}\tfrac{1}{(D+1)}\left(x^2 + 2\sin x\right) \\ &= \tfrac{1}{2}e^{-3x}\left(\tfrac{1}{(D+1)}x^2 + 2\tfrac{1}{(D+1)}\sin x\right) = \tfrac{1}{2}e^{-3x}\left(\left(1-D+D^2\right)x^2 + 2\tfrac{D-1}{D^2-1}\sin x\right) \\ &= \tfrac{1}{2}e^{-3x}\left(x^2 - 2x + 2 - (D-1)\sin x\right) = \tfrac{1}{2}e^{-3x}\left(x^2 - 2x + 2 - \cos x + \sin x\right) \end{aligned}$$

6. (a) If x^r is a solution to the associated homogeneous equation, then upon substitution into the equation, r must satisfy

$$\begin{aligned} (r(r-1)+3r-3) &= 0 \qquad \text{or} \quad r^2+2r-3 = (r-1)(r+3) = 0 \quad \text{so } r = 1,-3 \\ y_1 &= x \quad \text{and} \quad y_2 = x^{-3} \text{ are two linearly independent solutions} \end{aligned}$$

(b) Put $W = \begin{vmatrix} x & x^{-3} \\ 1 & -3x^{-4} \end{vmatrix} = -4x^{-3}$. Using the method of variation of parameters, a particular solution is $y_p = Ax + Bx^{-3}$, where

$$\begin{aligned} A' &= \tfrac{1}{W}\begin{vmatrix} 0 & x^{-3} \\ \frac{12}{1+x^2} & -3x^{-4} \end{vmatrix} = 3\tfrac{1}{1+x^2} \Rightarrow A = 3\arctan x \\ B' &= \tfrac{1}{W}\begin{vmatrix} x & 0 \\ 1 & \frac{12}{1+x^2} \end{vmatrix} = -3x^4\tfrac{1}{1+x^2} \Rightarrow B = -x^3 + 3x - 3\arctan x \\ y_p &= 3x\arctan x - 1 + 3x^{-2} - 3x^{-3}\arctan x = -1 + 3x^{-2} + 3\left(x - x^{-3}\right)\arctan x \\ y(x) &= -1 + \tfrac{3}{x^2} + 3\left(x - x^{-3}\right)\arctan x + c_1x + c_2x^{-3} \quad c_1 \text{ and } c_2 \text{ are arbitrary constants} \end{aligned}$$

7. We have

$$L\left\{te^{-2t}\sin 3t\right\} = -\frac{d}{ds}L\left\{e^{-2t}\sin 3t\right\} = -\frac{d}{ds}\;L\left\{\sin 3t\right\}|_{s\to s+2} = -\frac{d}{ds}\left(\tfrac{3}{(s+2)^2+9}\right) = \tfrac{6(s+2)}{\left((s+2)^2+9\right)^2}$$

8. Put $Y(s) = L\{y\}$, write $g(x) = \alpha\left(x-\frac{\pi}{2}\right)\sin x$, and apply the Laplace transform to the differential equation to obtain

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = L\left\{\alpha\left(x-\tfrac{\pi}{2}\right)\sin x\right\} = e^{-\frac{\pi}{2}s}L\left\{\sin(x+\tfrac{\pi}{2})\right\} = e^{-\frac{\pi}{2}s}L\left\{\cos x\right\} = \tfrac{s}{s^2+1}e^{-\frac{\pi}{2}s}$$

So

$$\begin{aligned} Y(s) &= \tfrac{s}{(s^2+1)(s^2+4)}e^{-\frac{\pi}{2}s} + \tfrac{s}{s^2+4} + \tfrac{2}{s^2+4} = \tfrac{1}{3}\left(\tfrac{s}{s^2+1} - \tfrac{s}{s^2+4}\right)e^{-\frac{\pi}{2}s} + \tfrac{s}{s^2+4} + \tfrac{2}{s^2+4} \\ y(x) &= \tfrac{1}{3}\alpha\left(x-\tfrac{\pi}{2}\right)\left(\cos(x-\tfrac{\pi}{2}) - \cos 2(x-\tfrac{\pi}{2})\right) + \cos 2x + \sin 2x \\ &= \tfrac{1}{3}\alpha\left(x-\tfrac{\pi}{2}\right)(\sin x + \cos 2x) + \cos 2x + \sin 2x = \begin{cases} \cos 2x + \sin 2x & x < \frac{\pi}{2} \\ \frac{1}{3}\sin x + \frac{4}{3}\cos 2x + \sin 2x & x \geq \frac{\pi}{2} \end{cases} \end{aligned}$$

9. Put $Y(s) = L\{y\}$ and apply the Laplace transform to the equation to obtain

$$\begin{aligned} Y(s) &= \tfrac{1}{s} + Y(s)\tfrac{s}{s^2+1} \Rightarrow Y(s) = \tfrac{s^2+1}{s(s^2-s+1)} = \tfrac{1}{s} + \tfrac{1}{s^2-s+1} \quad \text{(Partial Fractions)} \\ y(t) &= L^{-1}\{\tfrac{1}{s}\} + L^{-1}\{\tfrac{1}{s^2-s+1}\} = 1 + L^{-1}\{\tfrac{1}{(s-\frac{1}{2})^2+\frac{3}{4}}\} = 1 + e^{\frac{1}{2}t}L^{-1}\{\tfrac{1}{s^2+\frac{3}{4}}\} \\ &= 1 + \tfrac{2}{\sqrt{3}}e^{\frac{1}{2}t}\sin\tfrac{\sqrt{3}}{2}t = 1 + \tfrac{2\sqrt{3}}{3}e^{\frac{1}{2}t}\sin\tfrac{\sqrt{3}}{2}t \end{aligned}$$

10. Let $y = \sum_{n=0}^{\infty} a_n x^n$. Differentiate the power series term-wise to obtain y' and y'', and plug them into the equation to get

$$\begin{aligned} (x^2-1)\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + 4x\sum_{n=1}^{\infty} a_n n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n &= 0 \\ \sum_{n=2}^{\infty} a_n n(n-1)x^n - \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + 4\sum_{n=1}^{\infty} a_n n x^n + 2\sum_{n=0}^{\infty} a_n x^n &= 0 \\ \sum_{n=2}^{\infty} a_n n(n-1)x^n - \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + 4\sum_{n=1}^{\infty} a_n n x^n + 2\sum_{n=0}^{\infty} a_n x^n &= 0 \\ \sum_{n=0}^{\infty}\left(a_n(n+2)(n+1) - a_{n+2}(n+2)(n+1)\right)x^n &= 0 \end{aligned}$$

Thus, $a_{n+2} = a_n$ for all n. This gives $a_{2n} = a_0$ and $a_{2n+1} = a_1$, where a_0 and a_1 are arbitrary. The general solution is

$$y = a_0\sum_{n=0}^{\infty} x^{2n} + a_1\sum_{n=0}^{\infty} x^{2n+1} = a_0\left(1 + x^2 + x^4 + \cdots\right) + a_1\left(x + x^3 + x^5 + \cdots\right) = \frac{a_0 + a_1 x}{1-x^2}, \quad |x| < 1$$

The intial conditions $y(0) = 4$ and $y'(0) = 1$ force the values $a_0 = 4$ and $a_1 = 1$ and the initial value problem has the solution

$$y = 4 + x + 4x^2 + x^3 + 4x^4 + \cdots, \quad |x| < 1$$