MATHEMATICS DEPARTMENT **KUWAIT UNIVERSITY**

Differential equations (Math-240)

Date: **July 21, 2010** **Second Examination** **Duration : 90 Minutes**

Calculators and mobile phones are not allowed

Answer all questions

1. Find the general solution of the homogeneous linear differential equation,

$$(D^5 - 27D^2)y = 0.$$ 5 Mark

2. If $y_1 = \sin(x^2)$ is one of the solution of the homogeneous linear differential equation,

$$xy'' - y' + 4x^3y = 0, \qquad x > 0.$$ 5 Mark

find the general solution of the equation.

3. Solve the following nonhomogeneous linear equation by using **variation of parameters** method,

$$(D+2)^2y = e^{-2x}\sec^2 x\tan x .$$ 5 Mark

4. Make use of inverse differential operators formula to solve the equation,

$$(D^3 + 3D^2 + 5D + 3)y = e^{-x}(\cos x - 12x^2).$$ 5 Mark

5. **a.** Determine whether the set of functions $\{1,\ \tan x,\ \sec x\}$ are linearly dependent or independent.

3 Mark

b. Show that if $f(x)$ and $g(x)$ are linearly independent set of solutions of the equation,

$$y'' + p(x)y' + q(x)y = 0,$$ 2 Mark

then $f + 3g$ and $f - 2g$ are also linearly independent set of solutions of the above equation.

Inverse differential operator formulas

$$\frac{1}{f(D)}(e^{ax}u(x)) = e^{ax}\frac{1}{f(D+a)}u(x) ,$$

$$\frac{1}{D^2+a^2}\sin ax = -\frac{x}{2a}\cos ax,$$

$$\frac{1}{D^2+a^2}\cos ax = \frac{x}{2a}\sin ax.$$

Math-240 **second mid-term** July 21, 2010

1. $(D^5 - 27D^2)y = 0$, Auxiliary equation $m^5 - 27m^2 = 0$, $m^2(m^3 - 27) = 0$,

$m^2(m-3)(m^2+3m+9) = 0$, Roots are $\left(0, 0, 3, -\frac{3}{2} \pm i\frac{3\sqrt{3}}{2}\right)$

$y_g = c_1 + c_2x + c_3e^{3x} + e^{-\frac{3}{2}x}\left(c_4\cos\left(\frac{3\sqrt{3}}{2}x\right) + c_5\sin\left(\frac{3\sqrt{3}}{2}x\right)\right)$

2. $xy'' - y' + 4x^3y = 0, \quad x > 0.. \quad \cdots\cdots\cdot \ (1)$

Let $y_g = v(x)\sin(x^2)$, $y_g' = v'\sin(x^2) + 2xv\cos(x^2)$

$y_g'' = v''\sin(x^2) + 4xv'\cos(x^2) + 2v(\cos(x^2) - 2x^2\sin(x^2))$

Substitute in (1)

$x[v''\sin(x^2) + 4xv'\cos(x^2) + 2v(\cos(x^2) - 2x^2\sin(x^2))] - (v'\sin(x^2) + 2xv\cos(x^2)) + 4x^3v\sin(x^2) = 0$

$xv''\sin(x^2) + (4x^2\cos(x^2) - \sin(x^2))v' = 0$, Let $w = v'$, $w' = v''$

$w' + (4x\cot(x^2) - \frac{1}{x})w = 0$, $\frac{dw}{w} = -(4x\cot(x^2) - \frac{1}{x})dx$, $\ln w + (2\ln|\sin(x^2)| - \ln|x|) = \ln c_1$

$\frac{w\sin^2(x^2)}{x} = c_1$, $w = v' = c_1x\csc^2(x^2)$, $v = -\frac{1}{2}c_1\cot(x^2) + c_2$

$y_g = v\sin(x^2) = \sin(x^2)\left[-\frac{1}{2}c_1\cot(x^2) + c_2\right] = -\frac{1}{2}c_1\cos(x^2) + c_2\sin(x^2)$

3. $(D+2)^2y = (D^2+4D+4)y = e^{-2x}\sec^2x\tan x. \quad \cdots\cdots\cdot \ (1)$

$\| \quad \|(D+2)^2y = 0$, Auxiliary equation $(m+2)^2 = 0, \Rightarrow y_c = c_1e^{-2x} + c_2xe^{-2x}$,

Let $y_p = A(x)e^{-2x} + B(x)xe^{-2x} \cdots\cdots\cdot \ (2)$,

$y_p' = A'e^{-2x} - 2Ae^{-2x} + B'xe^{-2x} + B(1-2x)e^{-2x}$, set $A'e^{-2x} + B'xe^{-2x} = 0, \cdots\cdots\cdot \ (3)$

$y_p' = -2Ae^{-2x} + B(1-2x)e^{-2x} \cdots\cdots\cdot \ (4)$

$y_p' = -2A'e^{-2x} + 4Ae^{-2x} + B'(1-2x)e^{-2x} + 4B(x-1)e^{-2x} \cdots\cdots\cdot \ (5)$

Substitute (5), (4), (2) into (1) $\Rightarrow -2A'e^{-2x} + B'(1-2x)e^{-2x} = e^{-2x}\sec^2x\tan x \cdots\cdots\cdot \ (6)$

$-2A'e^{-2x} + B'(1-2x)e^{-2x} = e^{-2x}\sec^2x\tan x \cdots\cdots\cdot \ (6)$

$A'e^{-2x} + B'xe^{-2x} = 0, \quad \cdots\cdots\cdot \ (3)$

Solve $B' = \sec^2x\tan x$, $A' = -x\sec^2x\tan x$, $B = \frac{1}{2}\tan^2x$, $A = -\frac{1}{2}(x\tan^2x - \tan x - x)$

$y_p = -\frac{1}{2}(x\tan^2x - \tan x - x)e^{-2x} + \frac{1}{2}xe^{-2x}\tan^2x$, $y_g = y_c + y_p$.

4. $(D^3 + 3D^2 + 5D + 3)y = e^{-x}(\cos x - 12x^2) \quad \cdots\cdots (1)$

$(D^3 + 3D^2 + 5D + 3)y = 0$, Auxiliary equation

$m^3 + 3m^2 + 5m + 3 = (m+1)(m^2+2m+3) = 0$, Roots $\left(-1, -1 \pm i\sqrt{2}\right)$

$y_c = c_1e^{-x} + e^{-x}\left(c_2\cos\sqrt{2}x + c_3\sin\sqrt{2}x\right)$

$$y_p = \frac{1}{D^3+3D^2+5D+3}e^{-x}(\cos x - 12x^2)$$

$$= e^{-x}\frac{1}{(D-1)^3 + 3(D-1)^2 + 5(D-1) + 3}(\cos x - 12x^2) = e^{-x}\frac{1}{D(D^2+2)}(\cos x - 12x^2)$$

$$= e^{-x}\frac{1}{(D^2+2)}(\sin x - 4x^3) = e^{-x}\left[\sin x - \left(\frac{1}{2} - \frac{1}{4}D^2\right)4x^3\right] = e^{-x}[\sin x - (2x^3 - 6x)]$$

5. **a.** $\{1, \tan x, \sec x\}$, $c_1 + c_2\tan x + c_3\sec x = 0$

$$W = \begin{vmatrix} 1 & \tan x & \sec x \\ 0 & \sec^2x & \sec x\tan x \\ 0 & 2\sec^2x\tan x & \sec^3x + \sec x\tan^2x \end{vmatrix} = \sec^2x(\sec^3x + \sec x\tan^2x) - 2\sec^3x\tan^2x$$

$= \sec^3x(\sec^2x - \tan^2x) = \sec^3x \neq 0$, $x \neq \left(n + \frac{1}{2}\right)\pi$, $n = 0, \pm1, \pm2, \cdots\cdot$

Therefore the functions are linearly independent

b. We have $f'' + p(x)f' + q(x)f = 0$, $g'' + p(x)g' + q(x)g = 0$

So that $(f+3g)'' + p(x)(f+3g)' + q(x)(f+3g) =$

$(f'' + p(x)f' + q(x)f) + 3(g'' + p(x)g' + q(x)g) = 0$

and $(f-2g)'' + p(x)(f-2g)' + q(x)(f-2g) =$
$(f'' + p(x)f' + q(x)f) - 2(g'' + p(x)g' + q(x)g) = 0$

So that $f+3g$ and $f-2g$ are solutions of the equation.

$$W(f,g) = \begin{pmatrix} f & g \\ f' & g' \end{pmatrix} = fg' - gf' \neq 0$$

$$W(f+3g, f-2g) = \begin{pmatrix} f+3g & f-2g \\ f'+3g' & f'-2g' \end{pmatrix} = 5(gf' - fg') \neq 0 \text{ because } W(f,g) \neq 0$$

Therefore $f+3g$ and $f-2g$ are linearly independent.