MATHEMATICS & COMPUTER SCIENCE DEPARTMENT **KUWAIT UNIVERSITY**

Differential equations (Math-240)

Date: July 6, 2010 **First Examination** **Duration : 90 Minutes**

Calculators and mobile phones are not allowed

Answer all questions

1. Find the particular solution of the following initial value problem,

$$(x^2 - 1)y' + 2xy - \cos x = 0, \qquad y(0) = -1.$$ 5 Mark

2. Solve the following differential equation,

$$(2x - y - 1)\,dx + (x - 2y + 1)\,dy = 0\ .$$ 5 Mark

3. Find the general solution of the differential equation,

$$y^2dx - x(y + x^2)dy = 0.$$ 5 Mark

4. Solve the following differential equation,

$$\cos 2y(\cos 2y - \sin x)dx - 2\cos x \sin 2y\, dy = 0.$$ 5 Mark

5. Find the orthogonal trajectories of the following families of curves,

a) $y \ln|cx| = 1,$ c is a parameter. 3 Mark

b) $\sin(xy + b) = 0,$ b is a parameter. 2 Mark

Math-240 **second mid-term** July 6, 2010

1. $(x^2-1)y'+2xy-\cos x=0.$, $y'+\frac{2x}{x^2-1}y=\frac{1}{x^2-1}\cos x$

$I=\exp\left(\int\frac{2x}{x^2-1}dx\right)=\exp(\ln|x^2-1|)=x^2-1, \Rightarrow (x^2-1)y'+2xy=\cos x,$

$\frac{d}{dx}((x^2-1)y)=\cos x, \quad (x^2-1)y=\sin x+C, \quad y(0)=-1\Rightarrow C=1$

So that the solution is $y=\frac{1}{x^2-1}\sin x+\frac{1}{x^2-1}$

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2. $(2x-y-1)\,dx+(x-2y+1)\,dy=0\,. \quad \bullet\bullet\bullet\bullet(1)$

The two lines $2x-y-1=0$ and $x-2y+1=0$ intersect and the point of intersection is $(1,1)$

Let $x=u+1$ and $y=v+1$ so that (1) become

$(2u-v)du+(u-2v)dv=0. \quad \bullet\bullet\bullet\bullet(2)$ (homogeneous coefficients)

Let $u=vt, \quad du=vdt+tdv,$ substitute into (2)

$(2vt-v)(vdt+tdv)+(vt-2v)dv=0$ or $(2t-1)vdt+2(t^2-1)dv=0,$

$\frac{2t-1}{(t^2-1)}dt+\frac{1}{v}dv=0\,, \quad \frac{2t}{(t^2-1)}dt-\frac{1}{2}\left(\frac{1}{(t-1}-\frac{1}{t+1)}\right)dt+\frac{1}{v}dv=0$

$\ln|t^2-1|-\frac{1}{2}(\ln|t-1|-\ln|t+1|)+\ln v=\ln c, \quad \ln\left|v(t^2-1)\sqrt{\frac{t+1}{t-1}}\right|=\ln c$

$v\left((\frac{u}{v})^2-1\right)\sqrt{\frac{u+v}{u-v}}=c,\Rightarrow (y-1)\left(\left(\frac{x-1}{y-1}\right)^2-1\right)\sqrt{\frac{x+y-2}{u-y}}=c$

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3. $y^2dx-x(y+x^2)dy=0. \quad . \quad \bullet\bullet\bullet\bullet(1)$

$\frac{\partial M}{\partial y}=2y, \quad \frac{\partial N}{\partial x}=-y-3x^2, \quad \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{3y+3x^2}{-x(y+x^2)}=-\frac{3}{x}$

$I=\exp\left(\int-\frac{3}{x}dx\right)=x^{-3}$ Multiplied (1) by I $x^{-3}y^2dx-(x^{-2}y+1)dy=0, \quad \bullet\bullet\bullet\bullet(2)$ Exact equation

$x^{-3}y^2dx-(x^{-2}y+1)dy=0, \quad \bullet\bullet\bullet\bullet(2)$ Exact equation

$\frac{\partial F}{\partial x}=x^{-3}y^2 \bullet\bullet\bullet\bullet(3) \qquad \frac{\partial F}{\partial y}=-x^{-2}y-1 \quad \bullet\bullet\bullet\bullet(4)$

Integrate (3) $F(x,y)=-\frac{1}{2}x^{-2}y^2+B(y) \quad \bullet\bullet\bullet\bullet(5)$

Differentiate (5) $\frac{\partial F}{\partial y}=-x^{-2}y+B'(y)=-x^{-2}y-1, \quad \Rightarrow B'(y)=-1, \quad B(y)=-y$

So that $F(x,y)=-\frac{1}{2}x^{-2}y^2-y=C$

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4. $\cos 2y(\cos 2y-\sin x)dx-2\cos x\sin 2y\,dy=0 \quad \bullet\bullet\bullet\bullet\bullet(1)$

Let $t=\cos 2y, \quad dt=-2\sin 2ydy$

$t(t-\sin x)dx+\cos xdt=0$ or $\frac{dt}{dx}-\tan x\ t=-t^2\sec x \quad \bullet\bullet\bullet\bullet\bullet(2)$ (Bernoulis)

$t^{-2}\frac{dt}{dx}-\tan x\ t^{-1}=-\sec x,$ Let $u=t^{-1}, \quad du=-t^{-2}dt$

$\frac{du}{dx}+\tan x\ u=\sec x$ (Linear in u) $I=\exp\left(\int\tan xdx\right)=\sec x$

$\frac{d}{dx}\left(u\ \sec x\right)=\sec^2 x, \quad \Rightarrow \quad u\ \sec x=\tan x+c,$ or $\sec 2y\ \sec x=\tan x+c$

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5. **a**) $y\ln|cx|=1,$ or $\ln c+\ln x=\frac{1}{y}\,, \quad \frac{1}{x}=-\frac{1}{y^2}y', \quad y'=\frac{dy}{dx}=-\frac{y^2}{x},$ (differential equation of given family.) , $-\frac{dx}{dy}=-\frac{y^2}{x}$ (differential equation of orthogonal family.)

So that orthogonal family is $\frac{1}{2}x^2-\frac{1}{3}y^3=B$

b) $\sin(xy+b)=0, \quad xy+b=\sin^{-1}(0)=0$

$xy'+y=0, \quad \Rightarrow \quad \frac{dy}{dx}=-\frac{y}{x}$ (differential equation of given family.)

$-\frac{dx}{dy}=-\frac{y}{x}$ (differential equation of orthogonal family.)

So that orthogonal family is $\frac{1}{2}x^2-\frac{1}{2}y^2=C$