Kuwait University	Math. 240	May 12, 2010
Dept. of Math. & Comp. Sc.	Second Midterm Exam	Duration 90 Minutes

Calculators and mobile phones are NOT allowed

Instructions.

(i) Read each question carefully, **(ii)** Show all your work in order to receive full credit.

1. Answer each of the following as **True** or **False**. Justify your answer.

a. **(3 Points)** The functions $f(x) = x^3$, $g(x) = x^2|x|$ and $h(x) = x^3 + 1$ are linearly dependent on $[-1, 1]$.

b. **(3 Points)** If $A = (x^2 D - 2)$ and $B = (3D + x^3)$, where $D = \frac{d}{dx}$, then $AB = BA$.

2. **(3 Points)** Determine the function $g(x)$ of the nonhomogeneous linear equation

$$y'' - 2y' + 2y = g(x),$$

given that $y_p = 2\cos x + \sin x$ is a particular solution of the differential equation.

3. **(5 Points)** Use the method of **variation of parameters** to find the general solution of the differential equation

$$y'' - 3y' + 2y = e^x + x.$$

4. **(5 Points)** Find the general solution of the differential equation

$$y'' - \left(\frac{2}{x}\right)y' + \left(\frac{2}{x^2}\right)y = \frac{x}{1 + x^2};\ x > 0$$

given that $y_1 = x^2$ is a solution of the corresponding homogeneous equation.

5. **(6 Points)** Use the method of **inverse differential operators** to solve the equation

$$(D^2 - 3D + 2)(D^2 + 4)y = e^x + 16\sin 2x.$$

Some Useful Formulas For Inverse Differential Operators

$$\frac{1}{f(D)}[e^{ax}u(x)] = e^{ax}\frac{1}{f(D+a)}u(x) \qquad \frac{1}{g(D)(D-a)^k}(e^{ax}) = \frac{x^k e^{ax}}{k!\,g(a)};\ g(a) \neq 0$$

$$\frac{1}{D^2 + b^2}\cos(bx) = \frac{x\sin(bx)}{2b} \qquad \frac{1}{D^2 + b^2}\sin(bx) = -\frac{x\cos(bx)}{2b}$$

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1. (a) False $f,\ g,\ h$ are linearly independent.

Let $c_1x^3 + c_2x^2|x| + c_3(x^3+1) = 0$, then for $x \in [-1,1]$ we choose $x = -1,\ 0,\ 1$

$-c_1 + c_2 = 0,\ c_3 = 0\ c_1 + c_2 + 2c_3 = 0 \Rightarrow c_1 = c_2 = c_3 = 0$

1. (b) False $AB \neq BA$.

$ABy = [3x^2D^2 + (x^5-6)D + (3x^4-2x^3)]y \Rightarrow \boxed{AB = 3x^2D^2 + (x^5-6)D + (3x^4-2x^3).}$

$BAy = [3x^2D^2 + (x^5+6x-6)D - 2x^3]y \Rightarrow \boxed{BA = 3x^2D^2 + (x^5+6x-6)D - 2x^3.}$

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2. $g(x) = (D^2-2D+2)(2\cos x + \sin x)$

$= D^2(2\cos x+\sin x) - 2D(2\cos x + \sin x) + 2(2\cos x + \sin x) = \boxed{5\sin x.}$

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3. $y = y_c + y_p$:

$(m-1)(m-2) = 0 \Rightarrow \boxed{y_c = ce^x + c_2e^{2x}.}$

$$y_p = Ae^x + Be^{2x} \Rightarrow \begin{cases} A'e^x + B'e^{2x} &= 0 \\ A'e^x + 2B'e^{2x} &= e^x + x \end{cases} \Rightarrow \begin{cases} A' + B'e^x &= 0 \\ A' + 2B'e^x &= 1 + xe^{-x} \end{cases}$$

$$\begin{cases} A' &= -1 - xe^{-x} \\ B' &= e^{-x} + xe^{-2x} \end{cases} \Rightarrow \begin{cases} A &= -x + xe^{-x} + e^{-x} \\ B &= -e^{-x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} \end{cases} \Rightarrow \boxed{y_p = \tfrac{1}{4}(2x+3) - (1+x)e^x.}$$

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4. $y = y_1u = x^2u,\quad y' = x^2u' + 2xu,\quad y'' = x^2u'' + 4xu' + 2u$

$$u'' + \frac{2}{x}u' = \frac{1}{x(1+x^2)},\qquad x^2u' = \int \frac{xdx}{1+x^2} + c_1 = \tfrac{1}{2}\ln(1+x^2) + c_1$$

$$u = \tfrac{1}{2}\int \frac{\ln(1+x^2)}{x^2}dx + c_1\int \frac{dx}{x^2} + c_2 = -\tfrac{1}{2x}\ln(1+x^2) + \tan^{-1}x - \frac{c_1}{x} + c_2$$

$$\boxed{y = x\left[k + c_2x - \ln\sqrt{1+x^2} + x\tan^{-1}x\right];.k = -c_1}$$

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5. $y = y_c + y_{p_1} + y_{p_2}$

$$(m-1)(m-2)(m^2+4) = 0 \Rightarrow \boxed{y_c = c_1 e^x + c_2 e^{2x} + c_3 \cos 2x + c_4 \sin 2x.}$$

$$y_{p_1} = \frac{1}{(D-1)(D-2)(D^2+4)}(e^x) = \frac{xe^x}{(-1)(5)(1!)} = \boxed{-\tfrac{1}{5}xe^x.}$$

$$y_{p_2} = \frac{1}{D^2+4}\,\frac{1}{D^2-3D+2}(16\sin 2x) = \frac{16}{40}\,\frac{1}{D^2+4}(3D-2)(\sin 2x)$$

$$= \frac{2}{5}\,\frac{1}{D^2+4}(6\cos 2x - 2\sin 2x) = \boxed{\frac{x}{5}(3\sin 2x + \cos 2x).}$$