Kuwait University	Math 240	June 13, 2010
Department of Mathematics	Final Examination	Duration: 120 minutes

Calculators, cell phones, and pagers are not allowed.
Answer all of the following questions. Read each question carefully.

1. Consider the differential equation $\frac{dy}{dx} = e^{2x} + (1 + 2e^x)y + y^2$. **(2 points each)**

   (a) Given that $y_1 = -e^x$ is a solution of the differential equation, use the substitution $y = y_1 + \frac{1}{v}$ to reduce the equation to $v' = -(v + 1)$.

   (b) Sove the differential equation in (a) to find a family of solutions of the given differential equation.

2. Find the general solution of the equation $\frac{y}{x^2}\frac{dy}{dx} + e^{2x^3+y^2} = 0$. **(3 points)**

3. Consider the differential equation $(D^3 - D^2 + D - 1)\,y = 8xe^x$.

   (a) Use the method of **inverse differential operators** to find a particular solution of the given equation. **(3 points)**

   (b) Determine the complementary function and write the general solution of the differential equation. **(2 points)**

4. The functions $y_1 = x^2$ and $y_2 = x^{-4}$ are two linearly independent solutions of the associated homogeneous differential equation of

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 8y = 6\ln x \qquad x > 0$$

   (a) Use the **Variation of Parameters** method to find a particular solution of the given equation. **(4 points)**

   (b) Find the general solution of the given equation. **(2 points)**

5. Find the following. **(5 points each)**

   (a) $L\left\{\frac{2\cos t - 2\cosh t}{t}\right\}$ (b) $L^{-1}\left\{\frac{1}{(s^2 + 6s + 13)^2}\right\}$

6. Use the Laplace transform method to solve the initial value problem **(5 points)**

$$y'' + 4y = \varphi(t) = \begin{cases} 0, & \text{if } 0 \le t < 1 \\ t, & \text{if } t \ge 1, \end{cases} \qquad y(0) = 0, \qquad y'(0) = 1.$$

7. Use the Convolution Theorem to solve the integral equation **(4 points)**

$$F(t) - \int_0^t \beta F(t - \beta)\, d\beta = t.$$

8. Given the equation $(1 + x^2)y'' - 2y = 0$, <u>**derive**</u> the power series solution

$$y = a_0 + a_0x^2 + a_1\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(2k+1)(2k-1)}x^{2k+1}$$

   where a_0 and a_1 are arbitrary constants. **(3 points)**

FORMULAS FOR INVERSE OPERATORS

$$\frac{1}{f(D)}[e^{ax}u(x)] = e^{ax}\frac{1}{f(D+a)}u(x) \qquad \frac{1}{g(D)(D-a)^k}(e^{ax}) = \frac{x^k e^{ax}}{k!\, g(a)};\ g(a) \neq 0$$

$$\frac{1}{D^2+b^2}\cos(bx) = \frac{x\sin(bx)}{2b} \qquad \frac{1}{D^2+b^2}\sin(bx) = -\frac{x\cos(bx)}{2b}$$

TABLE FOR LAPLACE TRANSFORMS

$F(t)$	$f(s)$	$F(t)$	$f(s)$
1	$\frac{1}{s}$	$\sin bt$	$\frac{b}{s^2+b^2}$
e^{at}	$\frac{1}{s-a}$	$\cos bt$	$\frac{s}{s^2+b^2}$
t^n	$\frac{n!}{s^{n+1}}$	$\sinh bt$	$\frac{b}{s^2-b^2}$
t^x	$\frac{\Gamma(x+1)}{s^{x+1}}$	$\cosh bt$	$\frac{s}{s^2-b^2}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$		

FORMULAS FOR LAPLACE TRANSFORMS

$$L\{F(t)\} = f(s) \qquad L\{t^n F(t)\} = (-1)^n f^{(n)}(s)$$

$$L\{F'(t)\} = sf(s) - F(0) \qquad L\{e^{at}F(t)\} = f(s-a)$$

$$L\{F''(t)\} = s^2 f(s) - sF(0) - F'(0) \qquad L\{F(t-c)\,\alpha(t-c)\} = e^{-cs}f(s)$$

$$L\{F'''(t)\} = s^3 f(s) - s^2F(0) - sF'(0) - F'(0) \qquad L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)\,du$$

$$L\left\{\int_0^t F(u)\,G(t-u)\,du\right\} = f(s)\,g(s) \qquad L\left\{\int_0^t F(u)\,du\right\} = \frac{f(s)}{s}$$

Final Examination Solution Key (Second Semester 2009/2010)

1. $-e^x - \frac{1}{v^2}v' = e^{2x} + (1+2e^x)(-e^x+\frac{1}{v}) + (-e^x+\frac{1}{v})^2 = e^{2x} - e^x + \frac{1}{v} + -2e^{2x} + \frac{2e^x}{v} + e^{2x} + \frac{1}{v^2} - \frac{2e^x}{v}.$

$$\begin{aligned} -\tfrac{1}{v^2}v' &= \tfrac{1}{v} + \tfrac{1}{v^2} \qquad \text{or} \qquad v' = -(v+1) \\ \tfrac{1}{v+1}dv &= -dx \Rightarrow \ln c^{-1}(v+1) = -x \qquad \text{or} \qquad v = ce^{-x} - 1 \\ y &= \left(\tfrac{1}{c-e^x} - 1\right)e^x \end{aligned}$$

2. Write the equation in the separable form: $ye^{-y^2}\dfrac{dy}{dx} = -x^2e^{2x^3}$. Thus, $e^{-y^2} = \frac{1}{3}e^{2x^3} + c.$

3. The exponential shift yields: $e^{-x}(D-1)(D^2+1)y = D(D^2+2D+2)(e^{-x}y) = 8x$. Thus

$$e^{-x}y_p = 4\tfrac{1}{D^2+2D+2}x^2 = 4\left(\tfrac{1}{2} - \tfrac{1}{2}D + \tfrac{1}{4}D^2\right)x^2 \Rightarrow y_p = \left(2x^2 - 4x + 2\right)e^x$$

The solution of the homogeneous Eq is $y_c = c_1e^x + c_2\cos x + c_2\sin x \quad c_1, c_2 \in \mathbb{R}.$

The general solution is $y = c_1e^x + c_2\cos x + c_2\sin x + (2x^2 - 4x + 2)e^x \quad c_1, c_2 \in \mathbb{R}.$

4. (a) Let $y_p = Ax^2 + Bx^{-4}$, where A and B satisfy, $A'x^2 + B'x^{-4} = 0$ and $2A'x - 4B'x^{-5} = 6x^{-2}\ln x$. Solve for A' and B' and get

$$\begin{aligned} A' &= x^{-3}\ln x \qquad B' = -x^3\ln x \\ A &= \int x^{-3}\ln x dx = -\tfrac{1}{2}\int (x^{-2})'\ln x dx = -\tfrac{1}{2}x^{-2}\ln x + \tfrac{1}{2}\int x^{-3}dx = -x^{-2}\left(\tfrac{1}{4} + \tfrac{1}{2}\ln x\right) \\ B &= -\int x^3\ln x dx = -\tfrac{1}{4}\int (x^4)'\ln x dx = -\tfrac{1}{4}x^4\left(\ln x - \tfrac{1}{4}\right) \\ y_p &= Ax^2 + Bx^{-4} = -\tfrac{3}{16} - \tfrac{3}{4}\ln x \end{aligned}$$

(b) The associated homogeneous DE is normal and linear on $(0, \infty)$; its solutions would be generated by any two linearly independent solutions. Hence, the complementary function is $y_c = c_1x^2 + c_2x^{-4}$, ($c_1, c_2 \in \mathbb{R}$ are arbitrary) and $y = y_p + y_c = c_1x^2 + c_2x^{-4} - \frac{3}{16} - \frac{3}{4}\ln x$ is the general solution of the given nonhomogeneous equation.

5. (a) Use the fact that $L\{tF(t)\} = -\dfrac{d}{ds}L\{F(t)\}$ and get

$$f(s) = L\left\{\frac{2\cos t - 2\cosh t}{t}\right\} \qquad f'(s) = -2L\{\cos t - \cosh t\} = -2\left\{\tfrac{s}{s^2+1} - \tfrac{s}{s^2-1}\right\}$$

$$f(s) = -\ln\left(\tfrac{s^2+1}{s^2-1}\right) + C \to 0 \qquad \text{as } s \to 0 \Rightarrow C = 0. \qquad L\left\{\tfrac{2\cos t - 2\cosh t}{t}\right\} = \ln\left(\tfrac{s^2-1}{s^2+1}\right)$$

(b) Complete the square in $s^2 + 6s + 13 = (s+3)^2 + 4$, then use the second shift theorem and get

$$\begin{aligned} F(t) &= L^{-1}\left\{\tfrac{1}{(s^2+6s+13)^2}\right\} = L^{-1}\left\{\tfrac{1}{((s+3)^2+4)^2}\right\} = e^{-3t}L^{-1}\left\{\tfrac{1}{(s^2+4)^2}\right\} \\ &= e^{-3t}L^{-1}\left\{\tfrac{1}{s^2+4}\right\} * L^{-1}\left\{\tfrac{1}{s^2+4}\right\} = \tfrac{e^{-3t}}{4}\{\sin 2t * \sin 2t\} \\ &= \tfrac{e^{-3t}}{4}\int_0^t (\sin 2\tau)(\sin 2(t-\tau))\,d\tau = \tfrac{e^{-3t}}{8}\int_0^t [\cos(4\tau - 2t) - \cos 2t]\,d\tau \\ &= e^{-3t}\left(\tfrac{1}{16}\sin 2t - \tfrac{1}{8}t\cos 2t\right) = \tfrac{1}{16}e^{-3t}\sin 2t - \tfrac{1}{8}te^{-3t}\cos 2t \end{aligned}$$

6. $\varphi(t) = t\alpha(t-1) = (t-1)\alpha(t-1) + \alpha(t-1)$. Put $y(s) = L\{y(t)\}$. Then

$s^2y(s) - sy(0) - y'(0) + 4y(s) = L\{(t-1)\alpha(t-1)\} + L\{\alpha(t-1)\} = e^{-s}\frac{1}{s^2} + e^{-s}\frac{1}{s}$

$(s^2+4)y(s) - 1 = e^{-s}\frac{1}{s^2} + e^{-s}\frac{1}{s} \Rightarrow y(s) = e^{-s}\left(\frac{1}{s^2(s^2+4)} + \frac{1}{s(s^2+4)}\right) + \frac{1}{s^2+4}$

$= \frac{1}{4}e^{-s}\left(\frac{1}{s^2} - \frac{1}{s^2+4} + \frac{1}{s} - \frac{s}{s^2+4}\right) + \frac{1}{s^2+4}$

$y(t) = L^{-1}e^{-s}\left(\frac{1}{s^2} - \frac{1}{s^2+4} + \frac{1}{s} - \frac{s}{s^2+4}\right) + L^{-1}\frac{1}{s^2+4} = (t - \frac{1}{2}\sin 2(t-1) - \cos 2(t-1))\alpha(t-1) + \frac{1}{2}\sin 2t$

7. Put $f(s) = L\{F(t)\}$. Then

$f(s) - \frac{f(s)}{s^2} = \frac{1}{s^2} \Rightarrow f(s) = \frac{1}{s^2-1} \Rightarrow F(t) = L^{-1}\left\{\frac{1}{s^2-1}\right\} = \frac{1}{2}L^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{s+1}\right\} = \frac{e^t - e^{-t}}{2} = \sinh t$

8. Let $y = \sum_{k=0}^{\infty} a_k x^k$ so that $y' = \sum_{k=1}^{\infty} k a_k x^{k-1}$ and $y'' = \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2}$. Plug these into the equation $(1+x^2)y'' - 2y = 0$, and get

$$\begin{aligned}
\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} + x^2\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} - 2\sum_{k=0}^{\infty} a_k x^k &= 0 \\
\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k + \sum_{k=2}^{\infty} k(k-1)a_k x^k - 2\sum_{k=0}^{\infty} a_k x^k &= 0 \\
2(a_2 - a_0) + (6a_3 - 2a_1)x + \sum_{k=2}^{\infty}\left[(k+2)(k+1)a_{k+2} + k(k-1)a_k - 2a_k\right]x^k &= 0
\end{aligned}$$

Thus

$$\begin{aligned}
a_2 &= a_0, \quad a_3 = \frac{a_1}{3}, \quad \text{and} \quad (k+2)(k+1)a_{k+2} + (k+1)(k-2)a_k = 0 \\
a_{k+2} &= -\tfrac{k-2}{k+2}a_k \quad k = 2,3,4,\ldots \\
a_k &= -\tfrac{k-4}{k}a_{k-2} \quad k = 4,5,6,\ldots \\
\text{Then } a_{2k} &= 0 \quad k = 2,3,4,\ldots \\
a_5 &= -\tfrac{1}{5}a_3 = -\tfrac{1}{5\cdot 3}a_1 \quad a_7 = -\tfrac{3}{7}a_5 = \tfrac{3}{7}\tfrac{1}{5\cdot 3}a_1 = \tfrac{1}{7}\tfrac{1}{5}a_1 \\
a_{2k+1} &= \tfrac{(-1)^{k+1}}{(2k+1)(2k-1)}a_1 \quad k = 1,2,3,\ldots \\
y &= a_0 + a_0x^2 + a_1\sum_{k=0}^{\infty}\tfrac{(-1)^{k+1}}{(2k+1)(2k-1)}x^{2k+1}
\end{aligned}$$