Kuwait University	Math. 240	December 10, 2009
Dept. of Math. & Comp. Sc.	Second Midterm Exam	Duration 90 Minutes

Calculators and mobile phones are not allowed
Answer the following questions.

1. (**5 Points**) Solve the linear homogeneous equation
$$\left(D^5 + 6D^3 + 9D\right) y = 0.$$

2. (**5 Points**) Use inverse differential operators formulas to find a particular solution of the equation
$$(D - 1)\left(D^3 + D^2 + 1\right) y = 6xe^x.$$

3. (**5 Points**) Use the method of inverse differential operators to find the general solution of the equation
$$\left(D^3 - 3D^2 + 3D - 2\right) y = 2e^x \sin 2x.$$

4. (**3 Points**) Use exponential shift to solve the equation
$$(D + a)^2 y = e^{-ax} \sec^2 x \tan x.$$

5. (**3 Points**) Verify the formula
$$\frac{1}{D^2 + 2D + 5}\left(e^{-x} \sin 2x\right) = -\frac{1}{4} x e^{-x} \cos 2x.$$

6. (**2 Points**) Show that if f and g are linearly dependent differential functions, then their Wronskian vanishes identically.

7. (**2 Points**) If the functions y_1 and y_2 are linearly independent solutions of the equation
$$y'' + p(x) y' + q(x) y = 0$$
then show that $y_3 = y_1 + y_2$ and $y_4 = y_1 - y_2$ also form a linearly independent set of solutions.

Some Useful Formulas For Inverse Differential Operators

$$\frac{1}{f(D)}\left[e^{ax} u(x)\right] = e^{ax} \frac{1}{f(D + a)} u(x)$$

$$\frac{1}{D^2 + b^2} \cos(bx) = \frac{x \sin(bx)}{2b} \qquad \frac{1}{D^2 + b^2} \sin(bx) = -\frac{x \cos(bx)}{2b}$$

Math-240 second mid-term December 10, 2009

Answer

1. $\left(D^5+6D^3+9D\right)y=0.$

Auxiliary equation is $m^5+6m^3+9m=0, \quad m\left(m^4+6m^2+9\right)=m\left(m^2+3\right)^2=0$

The roots are $\left(0, \ \pm i\sqrt{3}, \ \pm i\sqrt{3}\right)$

The solution is $y=C_1+C_2\cos\sqrt{3}x+C_3\sin\sqrt{3}x+x\left(C_4\cos\sqrt{3}x+C_5\sin\sqrt{3}x\right)$

2. $(D-1)(D^3+D^2+1)y=6xe^x \quad y_p=\dfrac{1}{(D-1)(D^3+D^2+1)}\left(xe^x\right)=e^x\dfrac{1}{D((D+1)^3+(D+1)^2+1)}(6x)$

$=e^x\dfrac{1}{D(D^3+4D^2+5D+3)}(6x)=e^x\dfrac{1}{(D^3+4D^2+5D+3)}\left(3x^2\right)$

$=e^x\left(\frac{1}{3}-\frac{5}{9}D+\frac{1}{3}D^2\right)\left(3x^2\right)=e^x\left(x^2-\frac{10}{3}x+2\right)$

3. $\left(D^3-3D^2+3D-2\right)y= \ 2e^x\sin 2x. \quad y_c=C_1e^{2x}+e^{\frac{1}{2}x}\left(C_2\cos\frac{\sqrt{3}}{2}x+C_3\sin\frac{\sqrt{3}}{2}x\right)$

$y_p=\dfrac{2}{(D^3-3D^2+3D-2)}\ e^x\sin 2x= \ e^x\dfrac{2}{\left((D+1)^3-3(D+1)^2+3(D+1)-2\right)}\sin 2x$

$= \ 2e^x\dfrac{1}{(D^3-1)}\sin 2x= \ 2e^x\dfrac{1}{(-4D-1)}\sin 2x= \ -2e^x\dfrac{4D-1}{(16D^2-1)}\sin 2x=\frac{2}{65}e^x\left(8\cos 2x-\sin 2x\right).$

4. $(D+a)^2y=e^{-ax}\sec^2 x\tan x, \qquad y_c=C_1e^{-ax}+C_2xe^{-ax}$

$e^{ax}\ (D+a)^2y_p=\sec^2 x\tan x, \implies D^2\left(e^{ax}\ y_p\right)=\sec^2 x\tan x,$

$D\left(e^{ax}\ y_p\right)=\sec^2 x\tan x=\frac{1}{2}\sec^2 x, \quad \implies e^{ax}\ y_p=\frac{1}{2}\tan x, \quad \implies \quad y_p=\frac{1}{2}e^{-ax}\tan x$

$y_g=y_c+y_p=C_1e^{-ax}+C_2xe^{-ax}+\frac{1}{2}e^{-ax}\tan x.$

5. Verify the formula, $\dfrac{1}{D^2+2D+5}\left(e^{-x}\sin(2x)\right)=-\dfrac{x}{4}e^{-x}\cos(2x).$

$\left(D^2+2D+5\right)\left(\dfrac{1}{D^2+2D+5}\left(e^{-x}\sin(2x)\right)\right)= \ \left(D^2+2D+5\right)\left(-\dfrac{x}{4}e^{-x}\cos(2x)\right)$

$=-\dfrac{1}{4}e^{-x}\ \left((D-1)^2+2(D-1)+5\right)\left(x\cos(2x)\right)$

$=-\dfrac{1}{4}e^{-x}\ \left(D^2+4\right)\left(x\cos(2x)\right)=-\dfrac{1}{4}e^{-x}\ \left[D\left(\cos 2x-2x\sin 2x\right)+4x\cos(2x)\right]$

$=-\dfrac{1}{4}e^{-x}\ \left[-2\sin 2x-2\sin 2x-4x\cos 2x+4x\cos(2x)\right]=e^{-x}\sin(2x)$

6. Since f and g are linearly dependent, that is,

$c_1f+c_2g=0, \quad c_1, \ c_2\neq 0 \implies f=kg$, where $k=-\dfrac{c_2}{c_1}$, and $f'=kg'$.

The Wronskian of f and g is $\quad W(f,g)=\begin{vmatrix} f & g \\ f' & g' \end{vmatrix}=\begin{vmatrix} kg & g \\ kg' & g' \end{vmatrix}=kgg'-kgg'=0.$

7. $y''+p(x)y'+q(x)y=0, \ y_3''+p(x)y_3'+q(x)y_3=(y_1+y_2)''+p(x)(y_1+y_2)'+q(x)(y_1+y_2)$

$=y_1''+p(x)y_1'+q(x)y_1+y_2''+p(x)y_2'+q(x)y_1=0+0=0,$

$y''+p(x)y'+q(x)y=0, \ y_4''+p(x)y_4'+q(x)y_4=(y_1-y_2)''+p(x)(y_1-y_2)'+q(x)(y_1-y_2)$

$=y_1''+p(x)y_1'+q(x)y_1-(y_2''+p(x)y_2'+q(x)y_1)=0-0=0,$

The Wronskian of the functions y_1 and y_2 is $W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=y_1y_2'-y_2y_1'\neq 0$

and $\quad y_3=y_1+y_2$ and $y_4=y_1-y_2$

$W(y_3,y_4)=\begin{vmatrix} y_3 & y_4 \\ y_3' & y_4' \end{vmatrix}=\begin{vmatrix} y_1+y_2 & y_1-y_2 \\ y_1'+y_2' & y_1'-y_2' \end{vmatrix}=$

$=(y_1+y_2)(y_1'-y_2')-(y_1-y_2)(y_1'+y_2')=2y_2y_1'-2y_1y_2'$

$=2(y_2y_1'-y_1y_2')=2W(y_1,y_2)\neq 0$

Therefore the set $y_3=y_1+y_2$ and $y_4=y_1-y_2$ are also linearly independent.