KUWAIT UNIVERSITY

Department of Mathematics & Computer Science

Math 240		17 Jan 2010
Intro Diff Equations	Final Exam	Time: 2 hours

Calculators and mobile phones are not allowed in the exam.

1. Given the equation $(x - x^2 y \ln y)\, y' + y = 0$, find the solution curve passing through the point $(2, 1)$. [5 pts]

2. Consider the equation $x\,y'' + 2\,y' + 4\,x\,y = 0$: [4 + 4 pts]

   (a) Given that $y_1 = \frac{\sin 2x}{x}$ is one solution, use reduction of order to find a second solution y_2 such that y_1 and y_2 are linearly independent.

   (b) Use the answer to part (a) to find a particular solution to the equation

   $$x\,y'' + 2\,y' + 4\,x\,y = 4\,.$$

3. Definitions and theory: [1 + 2 + 3 pts]

   (a) What does it mean to say that a function $F(t)$ is of exponential order as $t \to \infty$?

   (b) Is the function $F(t) = \tan t$ sectionally continuous over the interval $[0, 3]$? Explain your answer.

   (c) Show that if $F(t)$ is of Class A, then $f'(s) = L\{-t\,F(t)\}$, where $f(s) = L\{F(t)\}$.

4. Given the function: [5 pts]

   $$F(t) = \begin{cases} 1\,, & 0 \le t < 1 \\ 1 - t\,, & 1 \le t < \pi \\ \sin t\,, & t \ge \pi \end{cases}$$

   Write $F(t)$ in terms of unit step functions then find its Laplace Transform.

5. Use Laplace transform to solve the initial value problem: [6 pts]

   $$y'' + 4y = 8\,e^{-2t}\,, \qquad y(0) = 2\,,\ y'(0) = 0\,.$$

6. Use the Laplace Transform method to solve for $F(t)$, where [5 pts]

   $$F(t) = e^{4t} + 6\int_0^t \cosh(4\beta)\,F(t-\beta)\,d\beta\,.$$

7. Find a power series solution to the equation [5 pts]

   $$(x^2 + 1)\,y'' + 3x\,y' + 8\,y = 0\,.$$

LT table →

ANSWERS

1. By inspection:

$$(x - x^2\, y\, \ln y)\, dy + y\, dx = \cdots = d(xy) - \frac{(xy)^2}{y} \ln y\, dy = 0 \quad\rightarrow\quad \frac{1}{(xy)^2}\, d(xy) - \frac{\ln y}{y}\, dy = 0 \quad\rightarrow$$

$$\frac{1}{xy} + (\ln y)^2 = c \quad\rightarrow\quad c = 1/2 \;\text{ if }\; y(2) = 1$$

2. (a) Put $y = v\, y_1 \;\rightarrow\; y' = v\, y_1' + y_1\, v' \;\rightarrow\; y'' = v\, y_1'' + 2v'\, y_1' + y_1\, v''$ and substitute in the equation to get

$$x\,(v\, y_1'' + 2v'\, y_1' + y_1\, v'') + 2\,(v\, y_1' + y_1\, v') + 4\, x\, v\, y_1 = 0 \quad\rightarrow$$

$$v'' + (4\cot 2x)\, v' = 0 \quad\rightarrow\quad v' = \csc^2 2x \quad\rightarrow\quad v = -\frac{\cot 2x}{2} \quad\rightarrow\quad y = -\frac{1}{2}\,\frac{\cos 2x}{x}$$

(b) First put the equation in the form $y'' + p(x)\, y' + q(x)\, y = R(x)$ which, in this case, is

$$y'' + \frac{2}{x}\, y' + 4\, y = \frac{4}{x}$$

Take $y_1 = (\sin 2x)/x$ and $y_2 = (\cos 2x)/x$ and find the Wronskian:

$$W(y_1, y_2) = y_1\, y_2' - y_2\, y_1' = \cdots = \frac{2}{x^2} \quad\rightarrow\quad \frac{1}{W(y_1, y_2)} = \frac{x^2}{2}$$

Accordingly,

$$y_p = -\frac{\cos 2x}{x} \int^x \frac{\sin 2t}{t}\,\frac{t^2}{2}\,\frac{4}{t}\, dt + \frac{\sin 2x}{x} \int^x \frac{\cos 2t}{t}\,\frac{t^2}{2}\,\frac{4}{t}\, dt = \cdots = \frac{1}{x}$$

3. Parts (a) and (c) in the book. The function $\tan t$ is not sectionally continuous over the given interval since it has an infinite discontinuity at $t = \pi/2 \in (0, 3)$.

4. In terms of unit step functions $F(t)$ becomes

$$\begin{aligned} F(t) &= 1 + \alpha(t-1)\,(1 - t - 1) + \alpha(t-\pi)\,(\sin t - 1 + t) \\ &= 1 - \alpha(t-1)\,\big((t-1) + 1\big) - \alpha(t-\pi)\,\big(\sin(t-\pi) - (t-\pi) - \pi + 1\big) \end{aligned}$$

Since $L\,[\alpha(t-c)\, G(t-c)] = \mathrm{e}^{-cs}\, g(s)$, it follows that

$$L[F(t)] = f(s) = \frac{1}{s} - \mathrm{e}^{-s}\left[\frac{1}{s^2} + \frac{1}{s}\right] - \mathrm{e}^{-\pi s}\left[\frac{1}{s^2+1} - \frac{1}{s^2} + \frac{1-\pi}{s}\right]$$

5. Based on the initial condition we have $L[y''] = s^2\, L[y] - 2s$ and the transform of the equation becomes

$$(s^2 + 4)\, L[y] = 2s + \frac{8}{s+2} \quad\rightarrow\quad L[y] = \frac{2s}{s^2+4} + \frac{8}{(s+2)\,(s^2+4)}$$

Use partial fractions to get

$$L[y] = \frac{s}{s^2+4} + \frac{2}{s^2+4} + \frac{1}{s+2} \quad\rightarrow\quad y = \cos 2t + \sin 2t + \mathrm{e}^{-2t}$$

6. The integral equation involves a convolution, that is

$$F(t) = \mathrm{e}^{4t} + 6\, G * F\,(t)$$

where $G(t) = \cosh 4t$. LT becomes

$$f(s) = \frac{1}{s-4} + 6\, g(s)\, f(s) = \frac{1}{s-4} + \frac{6s}{s^2-16}\, f(s) \quad\rightarrow\quad f(s) = \frac{1}{5}\left[\frac{6}{s-8} - \frac{1}{s+2}\right]$$

Take the inverse transform to get $F(t) = \big(6\,\mathrm{e}^{8t} - \mathrm{e}^{-2t}\big)/5$.

7. Start with $y = a_0 + a_1 x + \sum_2 a_n x^n$ and substitute in the equation

$$(x^2+1)\sum_2 n(n-1)\,a_n\,x^{n-2} + 3\,a_1\,x + \sum_2 3\,n\,a_n\,x^n + \sum_0 8\,a_n\,x^n = 0$$

$$\sum_2 n(n-1)\,a_n\,x^{n-2} + \sum_2 n(n-1)\,a_n\,x^n + 3\,a_1\,x + \sum_2 3\,n\,a_n\,x^n + \sum_0 8\,a_n\,x^n = 0$$

$$\sum_0 (n+2)(n+1)\,a_{n+2}\,x^n + \sum_2 n(n-1)\,a_n\,x^n + 3\,a_1\,x + \sum_2 3\,n\,a_n\,x^n + \sum_0 8\,a_n\,x^n = 0$$

$$(2\,a_2 + 8\,a_0) + (6\,a_3 + 11\,a_1)\,x + \sum_2 \Big[(n+2)(n+1)\,a_{n+2} + (n^2+2n+8)\,a_n\Big]\,x^n = 0$$

Equate the coefficients to zero to get (for $n > 1$)

$$a_n = -\frac{n^2-2n+8}{n(n-1)}\,a_{n-2} \quad\longrightarrow\quad \lim_{n\to\infty}\frac{a_n}{a_{n-2}} = -1$$