Kuwait University
Faculty of Science
Dept. of Math. & Comp. Sci.
Second Mid-Term Exam. in Ordinary differential equations (Math. 240)

Time: 90 min. Dec. 20, 2005.

Answer the following questions

1.(2 pts.) Answer the following as True or False and justify your answer.

(a) The function e^{t^2} is of exponential order.

(b) If $F(t)$ is not of class A, then $L\{F(t)\}$ does not exist.

2.(4 pts.) Show that x^{-2} is a solution for the equation

$$x^2y'' + xy' - 4y = 0.$$

Find a particular solution for the nonhomogeneous equation

$$x^2y'' + xy' - 4y = x^4.$$

3.(4 pts.) Use variation of parameters method to find the particular solution of

$$\frac{d^2y}{dx^2} + y = csc^2x \ \cot x$$

4.(8 pts.) Find the general solution of the following differential equations

(a) $(D^3 + D^2 - 5D + 3)y = e^{-x} \ \sin 2x$

(b) $(D^2 + 2D + 2)y = x^2$

5.(7 pts.) Evaluate Laplace transform for the following functions

(a) $F(t) = \int_0^t u \ \sin^2 u \ du$

(b) $G(t) = \cosh t \sinh t$

Some Laplace formulas

1. $L\{F'(t)\} = sL\{F(t)\} + F(0)$,
2. $L\{e^{kt}\} = \frac{1}{s-k}, \quad s > k$.
3. $L\{\sin kt\} = \frac{k}{s^2+k^2}$.
4. $L\{\cos kt\} = \frac{s}{s^2+k^2}$.
5. $L\{\sinh kt\} = \frac{k}{s^2-k^2} \quad |s| > |k|$.
6. $L\{\cosh kt\} = \frac{s}{s^2-k^2} \quad |s| > |k|$.

Key Solutions of the second Exam., Math. 240, Dec. 2005

1a F, $\lim_{t\to\infty} e^{-bt}e^{t^2} = \infty$. $\frac{1}{2}$ + $\frac{1}{2}$

1b. F, the function $t^{-1/2}$ is is not of class A, but its Laplâce exists. $\frac{1}{2}$ + $\frac{1}{2}$

2. $6x^{-4}x^2 - 2xx^{-3} - 4x^{-2} = 0$. Hence, x^{-2} is a solution for the given hom. equation. 1
Let $y = vx^{-2}$, $y' = v'x^{-2} - 2vx^{-3}$, $y'' = v''x^{-2} - 4v'x^{-3} + 6vx^{-4}$. Hence,
$x^2(v''x^{-2} - 4v'x^{-3} + 6vx^{-4}) + x(v'x^{-2} - 2vx^{-3}) - 4vx^{-2} = x^4$. Gives $v'' - 3x^{-1}v' = x^4$. $1\frac{1}{2}$
The I.F $= x^{-3}$. Hence, $(v'x^{-3})' = x$. implies $v' = \frac{x^5}{2}$,
$v = \frac{x^6}{12}$. Thus, $y_p = \frac{x^4}{12}$. $1\frac{1}{2}$

3. $B'(x) = -\csc x\ \cot x$, $A'(x) = \csc x\ \cot^2 x$. Hence, 2
$B(x) = \csc x$, $A(x) = -\frac{1}{2}(\csc x\ cotx + \ln|\csc x - \cot x|)$, (by parts)
$y_p = A(x)\sin x + B(x)\cos x$. 2

4a. $y_c = (c_1 + c_2x)e^x + c_3e^{-3x}$ 1
$y_p == \frac{1}{D^3+D^2-5D+3}\{e^{-x}\ \sin 2x\} = \frac{e^{-x}}{D^3-2D^2-4D+8}\{\sin 2x\} = \frac{e^{-x}}{16-8D}\{\sin 2x\} =$ 1
$\frac{e^{-x}}{8}\frac{2+D}{4-D^2}\{\sin 2x\} = \frac{e^{-x}}{32}(\sin 2x + \cos 2x)$. 2
$y_g = y_c + y_p$.

4b. $y_c = e^{-x}(c_1\cos 2x + c_2\sin 2x)$ 1
$y_p = \frac{1}{D^2+2D+2}\{x^2\} = \frac{1}{2}(1 + \frac{D^2+2D}{2})^{-1}\{x^2\} = \frac{1}{2}(1 - \frac{D^2+2D}{2} + (\frac{D^2+2D}{2})^2 + \cdots)\{x^2\}$ 2
$= \frac{1}{2}(x^2 - 2x + 1)$ 1
$y_g = y_c + y_p$.

5a. $F'(t) = t\sin^2 t$ (Fundamental th. of calculus).
$L\{\sin^2 t\} = L\{\frac{1-\cos 2t}{2}\} = \frac{1}{2}(\frac{1}{s} - \frac{s}{s^2+4})$ 1
$L\{t\sin^2 t\} = -\frac{d}{ds}L\{\sin^2 t\} = \frac{1}{2}(-\frac{1}{s^2} + \frac{4-s^2}{(s^2+4)^2})$ $1\frac{1}{2}$
$L\{F(t)\} = \frac{1}{2s}(-\frac{1}{s^2} + \frac{4-s^2}{(s^2+4)^2})$ (use the first formula of the given sheet) $1\frac{1}{2}$

5b. $L\{\cosh t\sinh t\} = \frac{1}{4}L\{e^{2t} - e^{-2t}\} = \frac{1}{4}(\frac{1}{s-2} - \frac{1}{s+2}) = \frac{1}{s^2-4}$.
1 2