Math 211 First Exam (Version A: Sections 1, 3, 51, and 53)

23 June 2018

Questions 1–5: 5 points each, questions 6-10: 15 points each.

1. The unit tangent vector to the curve $\mathbf{r}(t) = (t^2 - 4t)\,\mathbf{i} + (t^3 - t + 2)\,\mathbf{j} + (t^3 + 1)\,\mathbf{k}$ at the point $(5, 2, 0)$ is

(a) $-\frac{6}{7}\,\mathbf{i} + \frac{2}{7}\,\mathbf{j} + \frac{3}{7}\,\mathbf{k}$

(b) $\frac{5}{\sqrt{29}}\,\mathbf{i} + \frac{2}{\sqrt{29}}\,\mathbf{j}$

(c) $\frac{6}{7}\,\mathbf{i} - \frac{2}{7}\,\mathbf{j} - \frac{3}{7}\,\mathbf{k}$

(d) $-\frac{2}{\sqrt{17}}\,\mathbf{i} + \frac{2}{\sqrt{17}}\,\mathbf{j} + \frac{3}{\sqrt{17}}\,\mathbf{k}$

2. The arc-length of the curve $\mathbf{r}(t) = -4\cos t\,\mathbf{i} + 2t\,\mathbf{j} + 4\sin t\,\mathbf{k}$ between the points $(-4, 0, 0)$ and $(4, 2\pi, 0)$ is

(a) π (b) $\pi/2$ (c) $\sqrt{5}\,\pi$ (d) $2\sqrt{5}\,\pi$

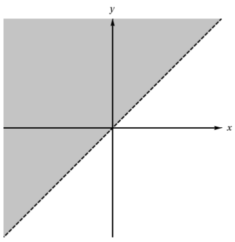
3. The curvature at any point of the curve $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 10\,\mathbf{k}$ is

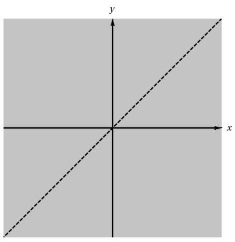
(a) 2 (b) 3 (c) 1/2 (d) 1/3

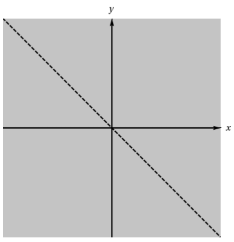
4. The limit of $f(x, y) = \dfrac{xy^2}{x^2 + y^4}$ as $(x, y) \to (0, 0)$ is

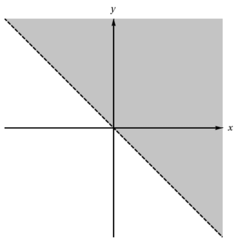
(a) 0 (b) 1/2 (c) 1 (d) Undefined

5. Which of the following regions is the domain of $f(x, y) = \dfrac{\cos(x + y)}{x - y}$?

(a) 

(b) 

(c) 

(d) 

6. Find the value and direction of the maximum directional derivative of $f(x, y) = xy^2 - 2\sqrt{x}$ at the point $(1, 2)$.

7. Find an equation for the tangent plane to the surface $x + y + z = e^{xyz}$ at the point $(3, 0, -2)$.

8. Let $z = f(x, y)$ where $x = s + t$, $y = s - t$, and f is a differentiable function. Show that

$$\left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2 = \frac{\partial z}{\partial s} \cdot \frac{\partial z}{\partial t}.$$

9. Find and classify all critical points of the function $f(x, y) = 4 + x^3 + y^3 - 3xy$.

10. Find the maximum value of $f(x, y, z) = xyz$ on the part of the paraboloid $x^2 + y^2 + z = 16$ that lies in the first octant.

Solutions

(Version A: Sections 1, 3, 51, and 53)

1. The correct choice is (a).

At the point $(5, 2, 0)$ we have $t^3 + 1 = 0$ which holds if and only if $t = -1$. On the other hand,

$$\mathbf{r}'(t) = (2t-4)\,\mathbf{i} + (3t^2-1)\,\mathbf{j} + 3t^2\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(-1) = -6\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k} \;\Rightarrow\; |\mathbf{r}'(-1)| = 7.$$

We conclude that $\mathbf{T}(-1) = -\frac{6}{7}\,\mathbf{i} + \frac{2}{7}\,\mathbf{j} + \frac{3}{7}\,\mathbf{k}$.

2. The correct choice is (d).

The points $(-4, 0, 0)$ and $(4, 2\pi, 0)$ correspond to $t = 0$ and $t = \pi$ respectively. On the other hand,

$$|\mathbf{r}'(t)| = |4\sin t\,\mathbf{i} + 2\,\mathbf{j} + 4\cos t\,\mathbf{k}| = 2\sqrt{5}.$$

Now the desired arc-length is

$$L = \int_0^\pi |\mathbf{r}'(t)|\,dt = \int_0^\pi 2\sqrt{5}\,dt = 2\sqrt{5}\,\pi.$$

3. The correct choice is (c).

The given curve is a circle of radius 2, hence it has curvature 1/2 at any point.

4. The correct choice is (d).

The limit is undefined by the two-path rule: along the line $y = 0$ we have

$$L_1 = \lim_{x\to 0} \frac{0}{x^2} = 0,$$

while along the parabola $x = y^2$ we get

$$L_2 = \lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2} \neq L_1.$$

5. The correct choice is (b).

The function is undefined only when the denominator equals 0. Hence $D_f = \big\{(x, y) \,\big|\, x - y \neq 0\big\}$.

6. $\nabla f = \langle y^2 - 1/\sqrt{x},\, 2xy\rangle \;\Rightarrow\; \nabla f(1, 2) = \langle 3, 4\rangle \;\Rightarrow\; |\nabla f(1, 2)| = 5.$

Hence the maximum directional derivative of f at the point $(1, 2)$ is 5 and it occurs in the direction of the vector $\langle 3, 4\rangle$.

7. Let $F(x, y, z) = x + y + z - e^{xyz}$. Then $\nabla F = \langle 1 - yz\,e^{xyz},\, 1 - xz\,e^{xyz},\, 1 - xy\,e^{xyz}\rangle$. Now the vector $\nabla F(3, 0, -2) = \langle 1, 7, 1\rangle$ is normal to the tangent plane at the point $(3, 0, -2)$. Therefore, an equation for this tangent plane is

$$1(x-3) + 7(y-0) + 1(z+2) = 0, \qquad \text{or} \qquad x + 7y + z = 1.$$

8. By the chain rule we have

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = f_x + f_y \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = f_x - f_y.$$

These imply $\dfrac{\partial z}{\partial s}\dfrac{\partial z}{\partial t} = (f_x + f_y)(f_x - f_y) = (f_x)^2 - (f_y)^2.$

9. The partial derivatives of f are $f_x = 3x^2 - 3y$ and $f_y = 3y^2 - 3x$ which are defined everywhere in $\mathbb{R}^2$. Thus the cirtical points of f are solutions of the system of equations $f_x = f_y = 0$, that is

$$\begin{cases} 3x^2 - 3y = 0 \\ 3y^2 - 3x = 0 \end{cases} \Rightarrow \begin{cases} y = x^2 \\ y^2 - x = 0 \end{cases} \Rightarrow x^4 - x = 0 \Rightarrow x(x^3 - 1) = 0 \Rightarrow x = 0, 1.$$

From the equation $y = x^2$ we see that the critical points of f are $(0,0)$ and $(1,1)$. We may classify these critical points using the Second Derivatives Test. With the discriminator

$$D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = (6x)(6y) - (-3)^2 = 9(4xy - 1)$$

we obtain:

- $D(0,0) = -9 < 0$, thus f has a saddle point at $(0,0)$.
- $D(1,1) = 27 > 0$ and $f_{xx}(1,1) = 6 > 0$, thus f has a local minimum at $(1,1)$.

10. Let $g(x,y,z) = x^2 + y^2 + z$. By the method of Lagrange multipliers the maximum occurs at a point on the paraboloid where $\nabla f = \lambda \nabla g$. That is a solution of the system

$$\begin{cases} yz = \lambda \cdot 2x \\ xz = \lambda \cdot 2y \\ xy = \lambda \cdot 1 \\ x^2 + y^2 + z = 16. \end{cases}$$

Note that in the first octant, $x, y, z > 0$. Thus $\lambda \neq 0$ since otherwise the first equation implies $y = 0$ or $z = 0$. Now multiplying the first equation by x, the second equation by y, and the third equation by z we obtain

$$xyz = 2\lambda x^2 = 2\lambda y^2 = \lambda z.$$

Since $\lambda \neq 0$ this gives $x^2 = y^2 = z/2$, which with the last equation implies $z/2 + z/2 + z = 16$. Therefore, $z = 8$ and $x^2 = y^2 = 4$. We conclude that the only solution of the above system in the first octant is the point $(2,2,8)$ which is where the maximum occurs. Hence the maximum value of f on the paraboloid $g = 16$ is $f(2,2,8) = 32$.

Math 211 First Exam (Version B: Sections 2, 52, and 54)

23 June 2018

Questions 1–5: 5 points each, questions 6-10: 15 points each.

1. The unit tangent vector to the curve $\mathbf{r}(t) = -(t^2+4t)\,\mathbf{i} + (t^3+3t)\,\mathbf{j} + (t^3+1)\,\mathbf{k}$ at the point $(3,2,0)$ is

(a) $\frac{2}{7}\,\mathbf{i} - \frac{6}{7}\,\mathbf{j} - \frac{3}{7}\,\mathbf{k}$

(b) $\frac{3}{\sqrt{13}}\,\mathbf{i} + \frac{2}{\sqrt{13}}\,\mathbf{j}$

(c) $-\frac{2}{7}\,\mathbf{i} + \frac{6}{7}\,\mathbf{j} + \frac{3}{7}\,\mathbf{k}$

(d) $-\frac{2}{3}\,\mathbf{i} + \frac{2}{3}\,\mathbf{j} + \frac{1}{3}\,\mathbf{k}$

2. The arc-length of the curve $\mathbf{r}(t) = -4\cos t\ \mathbf{i} + 2t\ \mathbf{j} + 4\sin t\ \mathbf{k}$ between the points $(-4,0,0)$ and $(0,\pi,4)$ is

(a) π (b) $\pi/2$ (c) $\sqrt{5}\,\pi$ (d) $2\sqrt{5}\,\pi$

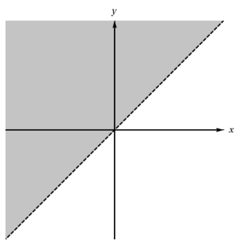
3. The curvature at any point of the curve $\mathbf{r}(t) = 3\cos t\ \mathbf{i} + 3\sin t\ \mathbf{j} + 10\ \mathbf{k}$ is

(a) 2 (b) 3 (c) 1/2 (d) 1/3

4. The limit of $f(x,y) = \dfrac{xy^2}{x^2+y^4}$ as $(x,y) \to (0,0)$ is

(a) 0 (b) 1/2 (c) 1 (d) Undefined

5. Which of the following regions is the domain of $f(x,y) = (y-x)\ln(y+x)$?

(a)


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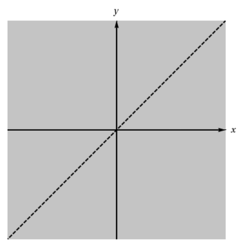
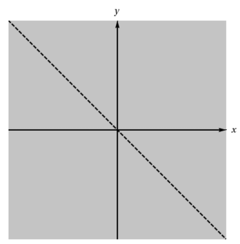
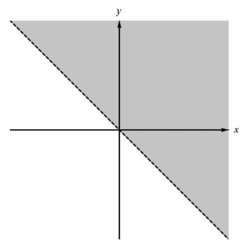


(c)


(d)


6. Find the value and direction of the maximum directional derivative of $f(x,y) = xy^2 - 2\sqrt{x}$ at the point $(1,1)$.

7. Find an equation for the tangent plane to the surface $x+y+z = \ln(1+xyz)$ at the point $(0,-2,2)$.

8. Let $w = f(u,v)$, where $u = xy$, $v = \dfrac{x}{y}$, and f is a differentiable function. Show that

$$x\,\frac{\partial w}{\partial x} + y\,\frac{\partial w}{\partial y} = 2xy\,\frac{\partial f}{\partial u}.$$

9. Find and classify all critical points of the function $f(x,y) = 6 - x^4 + 2x^2 - y^2$.

10. Find the maximum value of $f(x,y,z) = xyz$ on the part of the paraboloid $x^2+y^2+z = 1$ that lies in the first octant.

Solutions

(Version B: Sections 2, 52, and 54)

1. The correct choice is (c).

At the point $(3,2,0)$ we have $t^3+1=0$ which holds if and only if $t=-1$. On the other hand,

$$\mathbf{r}'(t) = -(2t+4)\,\mathbf{i} + (3t^2+3)\,\mathbf{j} + 3t^2\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(-1) = -2\,\mathbf{i} + 6\,\mathbf{j} + 3\,\mathbf{k} \;\Rightarrow\; |\mathbf{r}'(-1)| = 7.$$

We conclude that $\mathbf{T}(-1) = -\frac{2}{7}\,\mathbf{i} + \frac{6}{7}\,\mathbf{j} + \frac{3}{7}\,\mathbf{k}$.

2. The correct choice is (c).

The points $(-4,0,0)$ and $(0,\pi,4)$ correspond to $t=0$ and $t=\pi/2$ respectively. On the other hand,

$$|\mathbf{r}'(t)| = |4\sin t\,\mathbf{i} + 2\,\mathbf{j} + 4\cos t\,\mathbf{k}| = 2\sqrt{5}.$$

Now the desired arc-length is

$$L = \int_0^{\pi/2} |\mathbf{r}'(t)|\,dt = \int_0^{\pi/2} 2\sqrt{5}\,dt = \sqrt{5}\,\pi.$$

3. The correct choice is (d).

The given curve is a circle of radius 3, hence it has curvature 1/3 at any point.

4. The correct choice is (d).

The limit is undefined by the two-path rule: along the line $y=0$ we have

$$L_1 = \lim_{x\to 0} \frac{0}{x^2} = 0,$$

while along the parabola $x=y^2$ we get

$$L_2 = \lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2} \neq L_1.$$

5. The correct choice is (d).

The function is defined if and only if the argument of ln is positive. Hence $D_f = \big\{(x,y)\,\big|\, y+x>0\big\}$.

6. $\nabla f = \langle y^2 - 1/\sqrt{x},\, 2xy\rangle \;\Rightarrow\; \nabla f(1,1) = \langle 0,2\rangle \;\Rightarrow\; |\nabla f(1,1)| = 2.$

Hence the maximum directional derivative of f at the point $(1,1)$ is 2 and it occurs in the direction of the vector $\langle 0,2\rangle$.

7. Let $F(x,y,z) = x+y+z-\ln(1+xyz)$. Then $\nabla F = \langle 1-\frac{yz}{1+xyz},\, 1-\frac{xz}{1+xyz},\, 1-\frac{xy}{1+xyz}\rangle$. Now the vector $\nabla F(0,-2,2) = \langle 5,1,1\rangle$ is normal to the tangent plane at the point $(0,-2,2)$. Therefore, an equation for this tangent plane is

$$5(x-0)+1(y+2)+1(z-2)=0, \qquad \text{or} \qquad 5x+y+z=2.$$

8. By the chain rule we have

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} = yf_u + \tfrac{1}{y}f_v \quad \text{and} \quad \frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} = xf_u - \tfrac{x}{y^2}f_v.$$

These imply $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = xyf_u + \frac{x}{y}f_v + xyf_u - \frac{x}{y}f_v = 2xyf_u.$

9. The partial derivatives of f are $f_x = -4x^3 + 4x$ and $f_y = -2y$ which are defined everywhere in $\mathbb{R}^2$. Thus the cirtical points of f are solutions of the system of equations $f_x = f_y = 0$, that is

$$\begin{cases} -4x^3 + 4x = 0 \\ -2y = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} 4x\,(x^2 - 1) = 0 \\ y = 0 \end{cases}$$

From the first equation, x is 0 or ± 1. Thus the critical points of f are $(0, 0)$ and $(\pm 1, 0)$. We may classify these critical points using the Second Derivatives Test. With the discriminator

$$D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = (-12x^2 + 4)(-2) - (0)^2 = 8(3x^2 - 1)$$

we obtain:

- $D(0, 0) = -8 < 0$, thus f has a saddle point at $(0, 0)$.
- $D(\pm 1, 0) = 16 > 0$ and $f_{yy}(\pm 1, 0) = -2 < 0$, thus f has a local maximum at $(\pm 1, 0)$.

10. Let $g(x, y, z) = x^2 + y^2 + z$. By the method of Lagrange multipliers the maximum occurs at a point on the paraboloid where $\nabla f = \lambda \nabla g$. That is a solution of the system

$$\begin{cases} yz = \lambda \cdot 2x \\ xz = \lambda \cdot 2y \\ xy = \lambda \cdot 1 \\ x^2 + y^2 + z = 1. \end{cases}$$

Note that in the first octant, $x, y, z > 0$. Thus $\lambda \neq 0$ since otherwise the first equation implies $y = 0$ or $z = 0$. Now multiplying the first equation by x, the second equation by y, and the third equation by z we obtain

$$xyz = 2\lambda x^2 = 2\lambda y^2 = \lambda z.$$

Since $\lambda \neq 0$ this gives $x^2 = y^2 = z/2$, which with the last equation implies $z/2 + z/2 + z = 1$. Therefore, $z = \frac{1}{2}$ and $x^2 = y^2 = \frac{1}{4}$. We conclude that the only solution of the above system in the first octant is the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ which is where the maximum occurs. Hence the maximum value of f on the paraboloid $g = 1$ is $f(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = \frac{1}{8}$.