Math 211 Second Exam (Version A: Sections 1, 53, 54)

7 July 2018

Question 1: 10 points, questions 2-7: 15 points each.

1. Let $f(x)$ be a positive continuous function on $[a,b]$ and let D be the region under the graph $y = f(x)$ and above the interval $[a,b]$. Show that $\iint\limits_D x\,dA = \int_a^b x f(x)\,dx$.

2. Evaluate $\displaystyle\int_0^2 \int_{2y}^4 y\big(1+x^3\big)^{-\frac{1}{2}}\,dx\,dy$.

3. Let E be the solid that lies inside the sphere $x^2+y^2+z^2 = 2z$ and outside the sphere $x^2+y^2+z^2 = 3$. Give a description of E in spherical coordinates.

4. Let E be the solid below the cone $z = \sqrt{x^2+y^2}$ and above the ring $\frac{1}{4} \leqslant x^2+y^2 \leqslant 1$ in the xy–plane. Find the mass of E if the density function is $\rho(x,y,z) = 5y^2$.

5. Evaluate $\displaystyle\int_C xe^y\,ds$ where C is the line segment from $P(1,0)$ to $Q(\frac{1}{4},1)$.

6. Let R be the region in the first quadrant that is bounded by the ellipse $9x^2+4y^2 = 1$. Evaluate $\iint\limits_R y\,dA$ by making an appropriate change of variables.

7. Find the area of the part of the surface $z = 16+2x-y^2$ that lies above the triangle with vertices $(0,0)$, $(-3,1)$, and $(1,1)$.

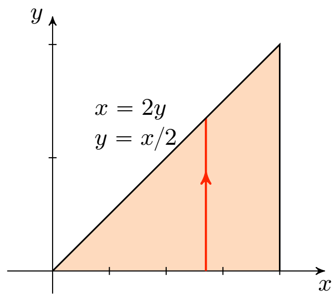
Solutions

1. The region D can be described as a Type I region: $D = \{(x,y) \mid a \leqslant x \leqslant b,\ 0 \leqslant y \leqslant f(x)\}$, which gives

$$\iint_D x\, dA = \int_a^b \int_0^{f(x)} x\, dy\, dx = \int_a^b x\Big[y\Big]_0^{f(x)} dx = \int_a^b x f(x)\, dx.$$

2. Note that $\displaystyle\int_0^2 \int_{2y}^4 y\big(1+x^3\big)^{-\frac12}\, dx\, dy = \iint_D y\big(1+x^3\big)^{-\frac12}\, dA$ where

$$D = \{(x,y) \mid 0 \leqslant y \leqslant 2,\ 2y \leqslant x \leqslant 4\}.$$



Using a sketch of D we may describe it as a Type II region:

$$D = \{(x,y) \mid 0 \leqslant x \leqslant 4,\ 0 \leqslant y \leqslant x/2\}$$

which gives

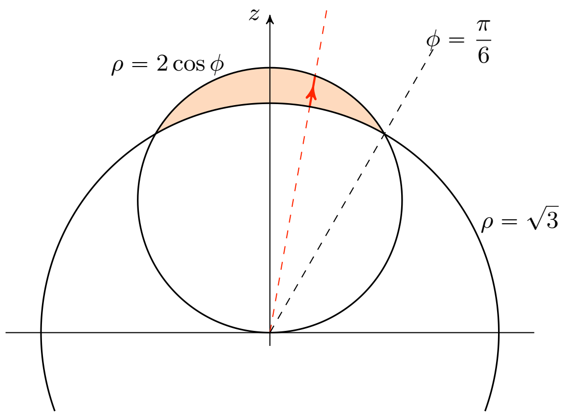
$$\iint_D y\big(1+x^3\big)^{-\frac12}\, dA = \int_0^4 \int_0^{x/2} y\big(1+x^3\big)^{-\frac12}\, dy\, dx = \int_0^4 \tfrac18 x^2\big(1+x^3\big)^{-\frac12}\, dx = \tfrac{1}{12}\big(1+x^3\big)^{\frac12}\Big]_0^4 = \tfrac{1}{12}\big(\sqrt{65}-1\big).$$

3. A sketch of a vertical cross section of the solid E is shown. The two given spheres are represented by the spherical equations $\rho = 2\cos\phi$ $(0 \leqslant \phi \leqslant \pi/2)$ and $\rho = \sqrt{3}$ respectively, thus their intersection is at

$$2\cos\phi = \sqrt{3} \ \Leftrightarrow\ \cos\phi = \frac{\sqrt{3}}{2} \ \Rightarrow\ \phi = \pi/6$$

Now a description of E is spherical coordinates is

$$E = \{(\rho,\theta,\phi) \mid 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \phi \leqslant \tfrac{\pi}{6},\ \sqrt{3} \leqslant \rho \leqslant 2\cos\phi\}.$$



4. In cylindrical coordinates, $E = \{(r,\theta,z) \mid 0 \leqslant \theta \leqslant 2\pi,\ \frac12 \leqslant r \leqslant 1,\ 0 \leqslant z \leqslant r\}$. Thus

$$m = \iiint_E 5y^2\, dV = \int_0^{2\pi} \int_{\frac12}^1 \int_0^r 5r^3 \sin^2\theta\, dz\, dr\, d\theta = \int_0^{2\pi} \int_{\frac12}^1 5r^4 \sin^2\theta\, dr\, d\theta$$

$$= \int_0^{2\pi} \Big[r^5\Big]_{\frac12}^1 \sin^2\theta\, d\theta = \frac{31}{32}\int_0^{2\pi} \tfrac12(1-\cos 2\theta)\, d\theta = \frac{31}{32}\left[\frac12\theta - \frac14\sin 2\theta\right]_0^{2\pi} = \frac{31\pi}{32}.$$

5. Using the vector $\overrightarrow{PQ} = \langle -\frac34, 1\rangle$, a parametric representation of the line segment C is

$$x(t) = 1 - \tfrac34 t, \quad y(t) = t, \quad 0 \leqslant t \leqslant 1,$$

using which we obtain

$$\int_C x e^y\, ds = \int_0^1 (1 - \tfrac34 t)\, e^t \sqrt{\big(-\tfrac34\big)^2 + 1^2}\, dt = \frac{5}{16}\int_0^1 (4-3t)\, e^t\, dt.$$

Now integrating by parts with $u = 4-3t$ and $dv = e^t dt$ gives

$$\int_C x e^y\, ds = \frac{5}{16}\Big[(4-3t)\, e^t + 3e^t\Big]_0^1 = \frac{5(4e-7)}{16}.$$

6. Consider the change of variables $u = 3x,\ v = 2y$. The region $R = \{(x,y) \mid (3x)^2 + (2y)^2 \leqslant 1,\ x, y \geqslant 0\}$ is the image under this transformation of the region

$$S = \{(u,v) \mid u^2 + v^2 \leqslant 1,\ u, v \geqslant 0\}$$

in the uv–plane. On the other hand, the Jacobian of this transformation is

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{6}.$$

These yield $\displaystyle\iint_R y\ dA = \iint_S \frac{1}{6}(v/2)\ du\,dv$. Note that S may be represented in polar coordinates by

$$0 \leqslant \theta \leqslant \pi/2, \quad 0 \leqslant r \leqslant 1,$$

which give

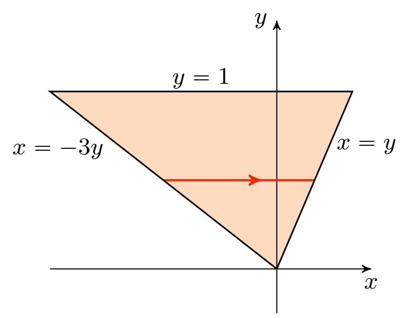
$$\iint_R y\ dA = \frac{1}{12}\int_0^{\pi/2}\int_0^1 r^2 \sin\theta\ dr\,d\theta = \frac{1}{36}\int_0^{\pi/2}\Big[r^3\Big]_0^1 \sin\theta\ d\theta = \frac{1}{36}\int_0^{\pi/2} \sin\theta\ d\theta = -\frac{1}{36}\cos\theta\Big]_0^{\pi/2} = \frac{1}{36}.$$

7. Let T be the region bounded by the given triangle. Then

$$T = \{(x,y) \mid 0 \leqslant y \leqslant 1,\ -3y \leqslant x \leqslant y\},$$

and we have

$$\begin{aligned} A(S) &= \iint_T \sqrt{1 + 2^2 + (-2y)^2}\ dA \\ &= \int_0^1 \int_{-3y}^{y} \sqrt{5 + 4y^2}\ dx\,dy = \int_0^1 4y\sqrt{5+4y^2}\ dx\,dy \\ &= \frac{1}{3}\left(5 + 4y^2\right)^{3/2}\Big]_0^1 = \frac{1}{3}\left(27 - 5\sqrt{5}\right). \end{aligned}$$



Math 211 Second Exam (Version B: Sections 2, 3, 51, 52)

7 July 2018

Question 1: 10 points, questions 2-7: 15 points each.

1. Let $f(x)$ be a positive continuous function on $[a, b]$ and let D be the region under the graph $y = f(x)$ and above the interval $[a, b]$. Show that $\displaystyle\iint_D y\,dA = \int_a^b \frac{1}{2}\big[f(x)\big]^2\,dx$.

2. Evaluate $\displaystyle\int_0^4 \int_{y/2}^2 y\big(1 + x^3\big)^{-\frac{1}{2}}\,dx\,dy$.

3. Let E be the solid that lies inside the sphere $x^2 + y^2 + z^2 = 6z$ and outside the sphere $x^2 + y^2 + z^2 = 9$. Give a description of E in spherical coordinates.

4. Let E be the solid below the cone $z = \sqrt{x^2 + y^2}$ and above the ring $1 \leqslant x^2 + y^2 \leqslant 4$ in the xy–plane. Find the mass of E if the density function is $\rho(x, y, z) = x^2$.

5. Evaluate $\displaystyle\int_C xe^y\,ds$ where C is the line segment from $P(\frac{1}{4}, 0)$ to $Q(1, 1)$.

6. Let R be the region in the first quadrant that is bounded by the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$. Evaluate $\displaystyle\iint_R x\,dA$ by making an appropriate change of variables.

7. Find the area of the part of the surface $z = 7 + 2x - \frac{1}{2}y^2$ that lies above the triangle with vertices $(0, 0)$, $(-1, 1)$, and $(3, 1)$.

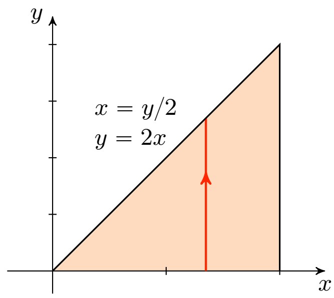
Solutions

1. The region D can be described as a Type I region: $D = \{(x,y) \mid a \leqslant x \leqslant b,\ 0 \leqslant y \leqslant f(x)\}$, which gives

$$\iint_D y\, dA = \int_a^b \int_0^{f(x)} y\, dy\, dx = \int_a^b \left[\tfrac{1}{2}y^2\right]_0^{f(x)} dx = \int_a^b \frac{1}{2}\big[f(x)\big]^2\, dx.$$

2. Note that $\displaystyle\int_0^4 \int_{y/2}^2 y(1+x^3)^{-\frac{1}{2}}\, dx\, dy = \iint_D y(1+x^3)^{-\frac{1}{2}}\, dA$ where

$$D = \{(x,y) \mid 0 \leqslant y \leqslant 4,\ \tfrac{y}{2} \leqslant x \leqslant 2\}.$$



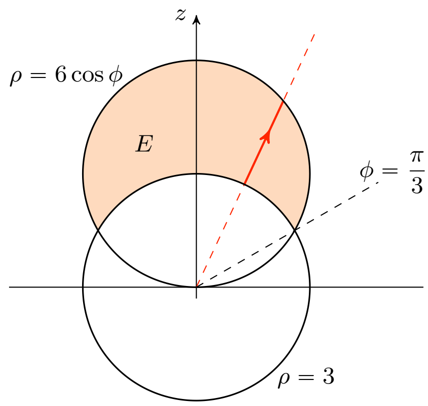
Using a sketch of D we may describe it as a Type II region:

$$D = \{(x,y) \mid 0 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant 2x\}$$

which gives

$$\iint_D y(1+x^3)^{-\frac{1}{2}}\, dA = \int_0^2 \int_0^{2x} y(1+x^3)^{-\frac{1}{2}}\, dy\, dx = \int_0^2 2x^2(1+x^3)^{-\frac{1}{2}}\, dx = \tfrac{4}{3}(1+x^3)^{\frac{1}{2}}\Big]_0^2 = \frac{8}{3}.$$

3. A sketch of a vertical cross section of the solid E is shown.



The two given spheres are represented by the spherical equations $\rho = 6\cos\phi$ $(0 \leqslant \phi \leqslant \pi/2)$ and $\rho = 3$ respectively, thus their intersection is at

$$6\cos\phi = 3 \;\Leftrightarrow\; \cos\phi = \tfrac{1}{2} \;\Rightarrow\; \phi = \pi/3$$

Now a description of E is spherical coordinates is

$$E = \{(\rho,\theta,\phi) \mid 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \phi \leqslant \tfrac{\pi}{3},\ 3 \leqslant \rho \leqslant 6\cos\phi\}.$$

4. In cylindrical coordinates, $E = \{(r,\theta,z) \mid 0 \leqslant \theta \leqslant 2\pi,\ 1 \leqslant r \leqslant 2,\ 0 \leqslant z \leqslant r\}$. Thus

$$m = \iiint_E x^2\, dV = \int_0^{2\pi} \int_1^2 \int_0^r r^3 \cos^2\theta\, dz\, dr\, d\theta = \int_0^{2\pi} \int_1^2 r^4 \cos^2\theta\, dr\, d\theta$$
$$= \int_0^{2\pi} \left[\tfrac{1}{5}r^5\right]_1^2 \cos^2\theta\, d\theta = \frac{31}{5}\int_0^{2\pi} \frac{1}{2}(1+\cos 2\theta)\, d\theta = \frac{31}{5}\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{31\pi}{5}.$$

5. Using the vector $\overrightarrow{PQ} = \langle \tfrac{3}{4}, 1\rangle$, a parametric representation of the line segment C is

$$x(t) = (1+3t)/4, \quad y(t) = t, \quad 0 \leqslant t \leqslant 1,$$

using which we obtain

$$\int_C xe^y\, ds = \frac{1}{4}\int_0^1 (1+3t)\, e^t \sqrt{\left(\tfrac{3}{4}\right)^2 + 1^2}\, dt = \frac{5}{16}\int_0^1 (1+3t)\, e^t\, dt.$$

Now integrating by parts with $u = 1+3t$ and $dv = e^t dt$ gives

$$\int_C xe^y\, ds = \frac{5}{16}\Big[(1+3t)\, e^t - 3e^t\Big]_0^1 = \frac{5(e+2)}{16}.$$

6. Consider the change of variables $u = x/3,\ v = y/2$. The region $R = \{(x,y) \,|\, (x/3)^2 + (y/2)^2 \leqslant 1,\ x, y \geqslant 0\}$ is the image under this transformation of the region

$$S = \{(u,v) \,|\, u^2 + v^2 \leqslant 1,\ u, v \geqslant 0\}$$

in the uv–plane. On the other hand, the Jacobian of this transformation is

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6.$$

These yield $\displaystyle\iint_R x\ dA = \iint_S 6(3u)\ du\,dv$. Note that S may be represented in polar coordinates by

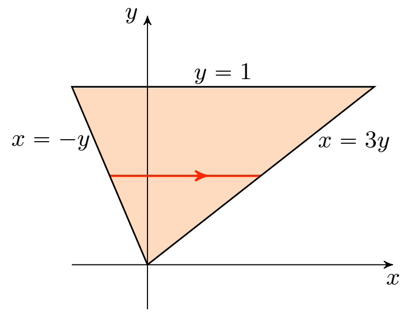
$$0 \leqslant \theta \leqslant \pi/2, \quad 0 \leqslant r \leqslant 1,$$

which give

$$\iint_R x\ dA = 6\int_0^{\pi/2}\int_0^1 3r^2\cos\theta\ dr\,d\theta = 6\int_0^{\pi/2}\Big[r^3\Big]_0^1 \cos\theta\ d\theta = 6\int_0^{\pi/2}\cos\theta\ d\theta = 6\sin\theta\Big]_0^{\pi/2} = 6.$$

7. Let T be the region bounded by the given triangle. Then

$$T = \{(x,y) \,|\, 0 \leqslant y \leqslant 1,\ -y \leqslant x \leqslant 3y\},$$



and we have

$$\begin{aligned} A(S) &= \iint_T \sqrt{1 + 2^2 + (-y)^2}\ dA \\ &= \int_0^1\int_{-y}^{3y}\sqrt{5+y^2}\ dx\,dy = \int_0^1 4y\sqrt{5+y^2}\ dx\,dy \\ &= \tfrac{4}{3}\left(5+y^2\right)^{3/2}\Big]_0^1 = \tfrac{4}{3}\big(6\sqrt{6} - 5\sqrt{5}\big). \end{aligned}$$