Math 211 Final Exam

14 May 2018

Questions 1–4: 10 points each, questions 5-8: 15 points each.

1. Use the Divergence Theorem to find the outward flux of the vector field

$$\mathbf{F}(x,y,z) = \left(x^3 - e^{yz}\right)\mathbf{i} + \left(y^3 + \sin z^2\right)\mathbf{j} + \left(z^3 - xy\right)\mathbf{k}$$

over the sphere $x^2 + y^2 + z^2 = z$.

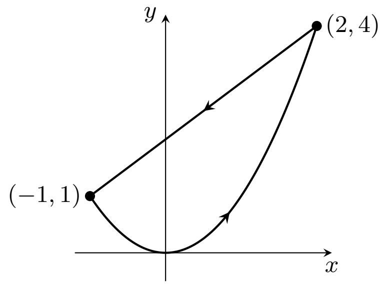
2. Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x\,y\,\sin x \cos y}{x^4 + y^2}$ or show that it does not exist.

3. (a) Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the paraboloid $z = x^2 + 2x + y^2$.

(b) Let C be the curve with vector equation $\mathbf{r}(t) = (\ln t)\,\mathbf{i} + t^2\,\mathbf{j} + 2\sqrt{t}\,\mathbf{k}$. Find an equation for the normal plane to C at the point $(0, 1, 2)$.

4. Let C be the curve which consists of the arc of the parabola $y = x^2$ from $(-1, 1)$ to $(2, 4)$, followed by the line segment from $(2, 4)$ to $(-1, 1)$. Use Green's Theorem to evaluate

$$\int_C \left(xy + \arctan x\right) dx + \left(x^2 + 3\ln(1+y)\right) dy.$$



5. Use Lagrange multipliers to find the absolute extrema of the function $f(x,y) = x^2 e^y$ on the circle $x^2 + y^2 = 3$.

6. Use the Fundamental Theorem for Line Integrals to evaluate

$$\int_C (yz+1)\,dx + (xz+1)\,dy + (xy+1)\,dz$$

where C is a smooth curve from $(-1, 2, 1)$ to $(2, 0, 1)$.

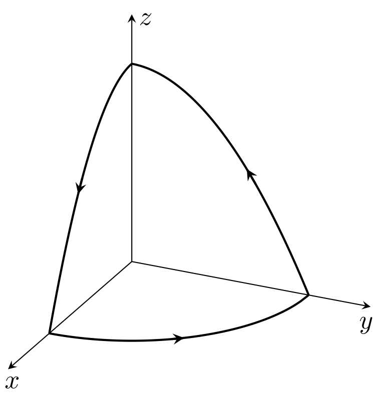
7. Evaluate $\displaystyle\iint_S y\,dS$ where S is the surface with parametric equations

$$x = u^2 - v, \quad y = u^2 + v, \quad z = 2v, \qquad 1 \leqslant u \leqslant 3, \quad -1 \leqslant v \leqslant 1.$$

8. Use Stokes' Theorem to evaluate $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ where

$$\mathbf{F}(x,y,z) = 2x^3 y\,\mathbf{i} + x^4\,\mathbf{j} + \left(yz + \sqrt{1+z^3}\right)\mathbf{k},$$

and C is the boundary of part of the paraboloid $z = 4 - x^2 - y^2$ that lies in the first octant, oriented as shown.



Solutions

1. Let S denote the given sphere, and E denote the solid region inside S. In spherical coordinates, the given sphere has equation $\rho = \cos\phi$ with $0 \leqslant \phi \leqslant \pi/2$. Thus

$$E = \{(\rho, \phi, \theta) \mid 0 \leqslant \rho \leqslant \cos\phi, \quad 0 \leqslant \phi \leqslant \pi/2, \ 0 \leqslant \theta \leqslant 2\pi\}.$$

Now by the Divergence Theorem we have

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \operatorname{div}\mathbf{F}\, dV = \iiint_E 3\left(x^2 + y^2 + z^2\right) dV \\ &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^{\cos\phi} 3\rho^2\rho^2 \sin\phi \ d\rho\, d\phi\, d\theta \\ &= \frac{\pi}{5}\int_0^{\pi/2} 6\cos^5\phi \sin\phi \ d\phi = \frac{\pi}{5}\Big[-\cos^6\phi\Big]_0^{\pi/2} = \frac{\pi}{5}.\end{aligned}$$

2. Along the x–axis the limit equals

$$L_1 = \lim_{x\to 0} \frac{0}{x^4} = 0,$$

and along the path $y = x^2$ it equals

$$L_2 = \lim_{x\to 0} \frac{x^3 \sin x \cos x^2}{2x^4} = \frac{1}{2}\lim_{x\to 0} \frac{\sin x}{x} \cdot \cos x^2 = \frac{1}{2}.$$

Since $L_1 \neq L_2$, by the two-path rule, the limit does not exist.

3. (a) $\mathbf{r}(t) = \langle \cos t, \sin t, 1 + 2\cos t\rangle$, $0 \leqslant t \leqslant 2\pi$.

(b) At the point $(0, 1, 2)$ on C, $\ln t = 0 \Leftrightarrow t = 1$. On the other hand, $\mathbf{r}'(t) = \langle 1/t, 2t, 1/\sqrt{t}\rangle$ is tangent to C, hence the normal plane at $(0, 1, 2)$ is normal to the vector $\mathbf{r}'(1) = \langle 1, 2, 1\rangle$. Therefore, an equation for the normal plane is $1(x-0)+2(y-1)+1(z-2) = 0$ which simplifies to $x + 2y + z = 4$.

4. Note that the line joining the points $(-1, 1)$ and $(2, 4)$ has the equation $y = x + 2$. Let

$$P(x, y) = xy + \arctan x, \qquad Q(x, y) = x^2 + 3\ln(1 + y),$$

and let D be the region bounded by the curve C. Then C is positively oriented and by Green's Theorem (verify that all its conditions are indeed satisfied) we obtain

$$\int_C P\, dx + Q\, dy = \iint_D (Q_x - P_y)\, dA = \iint_D (2x - x)\, dA = \int_{-1}^{2} \int_{x^2}^{x+2} x \ dy\, dx = \cdots = \frac{9}{4}.$$

5. Let $g(x, y) = x^2 + y^2$. By the method of Lagrange multipliers, the absolute extrema of f on the given circle occur at solutions of the system $\nabla f(x, y) = \lambda \nabla g(x, y)$, $g(x, y) = 3$, that is

$$\begin{cases} 2xe^y = \lambda(2x) \\ x^2 e^y = \lambda(2y) \\ x^2 + y^2 = 3. \end{cases}$$

Multiplying first equation by y and the second equation by x, then subtracting the two resulting equations we obtain $xe^y(2y - x^2) = 0$, which gives $x = 0$ or $x^2 = 2y$.

- If $x = 0$, the last equation gives $y = \pm\sqrt{3}$.
- If $2y = x^2$, the last equation gives $2y + y^2 = 3$ which holds if and only if $y = -3$ or $y = 1$. Now $y = -3$ is rejected since it yields $x^2 = -6$, while $y = 1$ gives $x = \pm\sqrt{2}$.

Since $f(0, \pm\sqrt{3}) = 0$ and $f(\pm\sqrt{2}, 1) = 2e$, we conclude that the minimum and maximum values of f on the circle $x^2 + y^2 = 3$ are 0 and $2e$ respectively.

6. Let f be a potential function of $\mathbf{F}$. Then $f_x = yz + 1$ which implies $f(x, y, z) = xyz + x + g(y, z)$ for some function g. Now since $f_y = xz + 1$, we must have $xz + 0 + g_y = xz + 1$, which implies $g(y, z) = y + h(z)$ for some function h. Finally, since $f_z = xy + 1$, we must have $xy + h'(z) = xy + 1$ which holds for $h(z) = z$. We conclude that $f(x, y, z) = xyz + x + y + z$ is a potential function for $\mathbf{F}$.

Now by the Fundamental Theorem for Line Integrals we obtain

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(2, 0, 1) - f(-1, 2, 1) = 3.$$

7. Let $\mathbf{r}(u, v) = \langle u^2 - v, u^2 + v, 2v \rangle$. Then

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u & 2u & 0 \\ -1 & 1 & 2 \end{vmatrix} = 2u \langle 2, -2, 2 \rangle \implies \|\mathbf{r}_u \times \mathbf{r}_v\| = 4\sqrt{3}\, u \qquad \text{(since } u \geqslant 0 \text{ on } S\text{)}.$$

Using this we obtain

$$\iint_S y \, dS = \int_1^3 \int_{-1}^1 (u^2 + v) \cdot 4\sqrt{3}\, u \; dv \, du = 4\sqrt{3} \int_1^3 \int_{-1}^1 (u^3 + uv) \; dv \, du = \cdots = 160\sqrt{3}.$$

8. Let S be part of the paraboloid $z = 4 - x^2 - y^2$ that lies in the first octant, oriented upward. Then C is the positively oriented boundary of S. By Stokes' Theorem (verify that all its conditions are satisfied) we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.$$

Note that

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x^3 y & x^4 & yz + \sqrt{1 + z^3} \end{vmatrix} = \langle z, \, 0, \, 2x^3 \rangle,$$

and that S is a graph $z = 4 - x^2 - y^2$ over the region

$$D = \{(x, y) \mid x^2 + y^2 \leqslant 4, \; x \geqslant 0, \; y \geqslant 0\} = \{(r, \theta) \mid 0 \leqslant r \leqslant 2, \; 0 \leqslant \theta \leqslant \pi/2\}.$$

Using these we obtain

$$\begin{aligned} \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \iint_D \Big[-(4 - x^2 - y^2)(-2x) - 0(-2y) + 2x^3 \Big] \, dA = \iint_D \big(8x - 2xy^2\big) \, dA \\ &= \int_0^{\pi/2} \int_0^2 \big(8r\cos\theta - 2r^3 \cos\theta \sin^2\theta\big) \, r \, dr \, d\theta = \cdots = \frac{64}{3} - \frac{64}{15} = \frac{256}{15}. \end{aligned}$$