Kuwait University	Math. 211	December 27, 2017
Dept.of Mathematics	Final Exam	Duration 120 Minutes

Calculators and Communication Devices are not allowed in the examination room

Multiple Choice Questions : Circle the correct answer. No justification is required.

(1) (5 Pts) The unit tangent vector at the point $(0,2,0)$ on the curve
$\mathbf{r}(t) = \left(\tan^{-1} t\right)\mathbf{i} + \left(2e^{2t}\right)\mathbf{j} + \left(8te^{t}\right)\mathbf{k}$ is

(A) $\mathbf{i}+4\mathbf{j}+8\mathbf{k}$ (B) $\mathbf{i}+2\mathbf{j}+8\mathbf{k}$ (C) $\mathbf{i}+4\mathbf{j}+16\mathbf{k}$

(D) $\frac{1}{9}(\mathbf{i}+4\mathbf{j}+8\mathbf{k})$ (E) $\frac{1}{9}(\mathbf{i}+2\mathbf{j}+8\mathbf{k})$

(D) $\mathbf{r}'(t)=\left(\frac{1}{1+t^2}\right)\mathbf{i}+\left(4e^{2t}\right)\mathbf{j}+8(1+t)e^t\mathbf{k}$: $\mathbf{T}(0)=\frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|}=\frac{1}{9}(\mathbf{i}+4\mathbf{j}+8\mathbf{k})$.

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(2) (5 Pts) The linearization of the function $f(x,y)=4+x\ln(xy-5)$ at the point $(2,3)$ is

(A) $6x+4y-24$ (B) $6x+4y+24$ (C) $6x+4y-20$

(D) $4x+6y+20$ (E) $3x+2y-20$

(C) $f(x,y)=4+x\ln(xy-5)$ $f_x=x\ln(xy-5)+\frac{xy}{xy-5}$ $f_y=\frac{x^2y}{xy-5}$

$L(x,y)=f(2,3)+f_x(2,3)(x-2)+f_y(2,3)(y-3)=6x+4y-20.$

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(3) (5 Pts) The direction of greatest rate of change of $w=3x^2-xy+z$ at the point $(1,-1,6)$ is

(A) $2\mathbf{i}-\mathbf{j}+\mathbf{k}$ (B) $3\mathbf{i}-2\mathbf{j}+\mathbf{k}$ (C) $3\mathbf{i}-\mathbf{j}+2\mathbf{k}$

(D) $7\mathbf{i}-\mathbf{j}+6\mathbf{k}$ (E) $7\mathbf{i}-\mathbf{j}+\mathbf{k}$

(E $\nabla w(x,y,z)=(6x-y)\mathbf{i}-x\mathbf{j}+\mathbf{k}$. $\nabla w(1,-1,6)=7\mathbf{i}-\mathbf{j}+\mathbf{k}$.

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(4) (5 Pts) If $3x^2y+z\cos y=xz^3$, then $\frac{\partial y}{\partial z}$ equals to

(A) $\frac{3x^2-z\sin y}{3xz^2-\cos y}$ (B) $\frac{3x^2+z\sin y}{3xz^2+\cos y}$ (C) $\frac{3xz^2-\cos y}{3x^2-z\sin y}$

(D) $\frac{6xy-z^3}{3xz^2-\cos y}$ (E) $\frac{6xy-z^3}{3x^2-z\sin y}$

(C) $f(x,y,z)=3x^2y+z\cos y-xz^3$: $\frac{\partial y}{\partial z}=-\frac{f_z}{f_y}=\frac{3xz^2-\cos y}{3x^2-z\sin y}$.

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(5) (5 Pts) The volume of the solid region bounded by the surfaces $y = x^2,\ y+z = 1$, and $z = 0$ can be found by evaluating

$$(A)\ \int_{-1}^{1}\int_{1}^{x^2}(1-y)\,dydx \qquad (B)\ \int_{-1}^{1}\int_{x^2}^{1}(1-y)\,dxdy \qquad (C)\ \int_{0}^{1}\int_{0}^{\sqrt{y}}(1-y)\,dxdy$$

$$(D)\ \int_{-1}^{1}\int_{x^2}^{1}(1-y)\,dydx \qquad (E)\ \int_{-1}^{1}\int_{x^2}^{1}(y-1)\,dydx$$

(D) $$V = \iint_D f(x,y)\,dA = \int_{-1}^{1}\int_{x^2}^{1}(1-y)\,dxdy.$$

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(6) (10 Pts) Find the area of the surface with parametric equations

$$x = u^2,\ y = uv,\ z = \frac{1}{2}v^2,\ 0 \le u \le 1,\ 0 \le v \le 2$$

Solution

The tangent vectors are $\mathbf{r}_u = 2u\mathbf{i} + v\mathbf{j}$, and $\mathbf{r}_v = u\mathbf{j} + v\mathbf{k}$. So, a normal vector to the surface is $\mathbf{r}_u \times \mathbf{r}_v = (2u\mathbf{i} + v\mathbf{j}) \times (u\mathbf{j} + v\mathbf{k}) = v^2\mathbf{i} - 2uv\mathbf{j} + 2u^2\mathbf{k}$

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\ dA = \int_0^1\int_0^2 \left(2u^2 + v^2\right) dvdu = \int_0^1 \left(4u^2 + \tfrac{8}{3}\right) du = \boxed{4}$$

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(7) (10 Pts) Show that $\mathbf{F} = (e^y)\,\mathbf{i} + (xe^y + e^z)\,\mathbf{j} + (ye^z)\,\mathbf{k}$ is a conservative vector field then use this fact to evaluate $\displaystyle\int_{(0,2,0)}^{(4,0,3)} \mathbf{F}\cdot d\mathbf{r}$.

Solution

We have curl $\mathbf{F} = \mathbf{O}$, the domain of $\mathbf{F}$ is $\mathbb{R}^3$, and the components of $\mathbf{F}$ have continuous partial derivatives, so $\mathbf{F}$ is conservative and thus its line integral is independent of path, that is

$$\int_A^B \mathbf{F}\cdot d\mathbf{r} = f(B) - f(A),$$ Where f is a potential function of $\mathbf{F}$.

From $f_x = e^y,\ f_y = xe^y + e^z$, and $f_z = ye^z$ we obtain $f(x,y,z) = xe^y + ye^z$ as a potential function of $\mathbf{F}$. Hence $\displaystyle\int_{(0,2,0)}^{(4,0,3)} \mathbf{F}\cdot d\mathbf{r} = f(4,0,3) - f(0,2,0) = 4 - 2 = \boxed{2}$

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(8) Use the Divergence Theorem to find the outward flux of the vector field

$$\mathbf{F} = \left(xz^2\right)\mathbf{i} + \left(x^2y - z^3\right)\mathbf{j} + \left(2xy + y^2z\right)\mathbf{k}$$

across the boundary surface of the solid region enclosed by the hemisphere $z = \sqrt{4 - x^2 - y^2}$ and the $xy-$plane.

Solution

$\iint\limits_S \mathbf{F}\cdot\mathbf{n}\, dS \overset{Div.\ Th.}{=} \iiint\limits_E \operatorname{div}\mathbf{F}\, dV$. We have $\operatorname{div}\mathbf{F} = z^2 + x^2 + y^2$, and

$E = \left\{(\rho,\phi,\theta) \mid 0 \le \rho \le 2,\ \ 0 \le \phi \le \frac{\pi}{2},\ 0 \le \theta \le 2\pi\right\}$

$$\iiint\limits_E \operatorname{div}\mathbf{F}\, dV = \iiint\limits_E \left(x^2 + y^2 + z^2\right)\, dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho^2\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\phi\, d\phi \int_0^2 \rho^4 d\rho = \boxed{\frac{64\pi}{5}}$$

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(9) (15 Pts) Verify Green's Theorem for $\mathbf{F} = xy\mathbf{i} + \left(x^2 + y^2\right)\mathbf{j}$, and C is the line segment from $(0,0)$ to $(1,1)$ followed by the arc of the curve $x = y^2$ from $(1,1)$ to $(0,0)$.

Solution

Let C_1 be the line segment from $(0,0)$ to $(1,1)$, .and C_2 be the arc of the curve $x = y^2$ from $(1,1)$ to $(0,0)$.

$$\oint\limits_C (xy)\, dx + \left(x^2 + y^2\right) dy = \int\limits_{C_1} (xy)\, dx + \left(x^2 + y^2\right) dy + \int\limits_{C_2} (xy)\, dx + \left(x^2 + y^2\right) dy$$

$$= \int_0^1 3y^2 dy + \int_1^0 \left(3y^4 + y^2\right) dy = 1 - \tfrac{14}{15} = \boxed{\tfrac{1}{15}}$$

Now, let D be the region enclosed by C

$$\iint\limits_D \left[\frac{\partial}{\partial x}\left(x^2 + y^2\right) - \frac{\partial}{\partial}(xy)\right] dA = \iint\limits_D x\,dA = \int_0^1\int_{y^2}^{y} x\,dx\,dy$$

$$= \int_0^1 \tfrac{1}{2}\left(y^2 - y^4\right) dy = \tfrac{1}{2}\left(\tfrac{1}{3} - \tfrac{1}{5}\right) = \boxed{\tfrac{1}{15}}$$

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(10) (15 Pts) Use Stokes's Theorem to evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where

$$\mathbf{F} = \left(x - \frac{1}{2}y^2\right)\mathbf{i} + \left(y - \frac{1}{2}z^2\right)\mathbf{j} + \left(z - x^2\right)\mathbf{k}$$

and C is the positively oriented triangle with vertices $(2,0,0)$, $(0,2,0)$ and $(0,0,4)$

Solution

$$\oint_C \mathbf{F}\cdot d\mathbf{r} \overset{St,\ Th.}{=} \iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\ dS.$$

$\operatorname{curl}\mathbf{F} = z\mathbf{i} + 2x\mathbf{j} + y\mathbf{k}$ and we take the surface S to be the planar region enclosed by C, so S is the portion of the plane $2x + 2y + z = 4$ over $D : 0 \le x \le 2,\ 0 \le y \le 2 - x$.

Since C is oriented counterclockwise, we orient S upward.

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\ dS = \iint_D \left(-P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R\right)\ dA, \text{ where } z = 4 - 2x - 2y$$

$$= \iint_D (2z + 4x + y)\ dA = \iint_D (8 - 3y)\ dA$$

$$= \int_0^2 \int_0^{2-x} (8 - 3y)\, dydx = \int_0^2 \left(10 - 2x - \tfrac{3}{2}x^2\right) dx = \boxed{12}$$

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(11) (15 Pts) Let R be the square with vertices $(0,2)$, $(1,1)$, $(2,2)$, and $(1,3)$.

Evaluate $\iint_R \left(\frac{y-x}{y+x}\right) dA$ by making an appropriate change of variables.

Solution

The terms $y - x$ and $y + x$ in the integrand suggests the new variables $u = y - x$ and $v = y + x$. Solving for x and y, we get $T : x = \frac{1}{2}(-u + v)$, $y = \frac{1}{2}(u + v)$.

The Jacobian of T is $J = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}$.

To find S corresponding to R we note that the sides of R lie on the lines: $y - x = 0$, $y - x = 2$, $y + x = 2$, and $y + x = 4$, so the image lines in $uv-$plane are: $u = 0$, $u = 2$, $v = 2$, and $v = 4$. Thus S is given by $S : 0 \le u \le 2,\ 2 \le v \le 4$. The integration now follows

$$\iint_R \left(\frac{y-x}{y+x}\right)\ dA = \int_0^2 \int_2^4 \left(\frac{u}{v}\right)\left|-\tfrac{1}{2}\right| dudv = \frac{1}{2}\int_0^2 udu \int_2^4 \frac{dv}{v} = \boxed{\ln 2}$$
