Kuwait University | *Math111. Linear Algebra* | **Date: June 30, 2018**
Department of Mathematics | *First Exam* | **Duration: 90 minutes**

Calculators and communication devices are not allowed.
Answer the following questions.

1. (a) Let $A = \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}$.

   (*i*) (5 pts.) Find A^{-1}.

   (*ii*) (5 pts.) Find $\operatorname{tr}(A) - \operatorname{tr}(A^{-1})$.

   (b) (5 pts.) Let A and B be two $n \times n$ matrices and $c \in \mathbb{R}$. Show that
   $$\operatorname{tr}(A + cB) = \operatorname{tr}(A) + c\operatorname{tr}(B).$$

   (c) (15 pts.) Find, if any, all value(s) of a for which the system
   $$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & 2 \\ 2x & + & 5y & + & 3z & = & 1 \\ x & + & 2y & - & az & = & a \end{array}$$

   has:

   (*i*) no solution (*ii*) unique solution (*iii*) infinitely many solutions.

2. (a) (10 pts.) Let A be an $n \times n$ matrix. Show that A is invertible if and only if, $\det(A) \neq 0$.

   (b) (10 pts.) Show that if a matrix has an inverse, then the inverse is unique.

   (c) (10 pts.) Prove that any square matrix A can be written as $A = S + K$, where S is symmetric and K is skew-symmetric.

3. (a) (10 pts.) Let B be a 3×3 matrix, where $\det(B) = -\frac{1}{2}$. Find $\det(\operatorname{adj}(2B))$.

   (b) Let $A^{-1} = \begin{bmatrix} 1 & -2 & 2 & 1 \\ 1 & 1 & -1 & 4 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 2 \end{bmatrix}$.

   (*i*) (5 pts.) Find $\det(A)$.

   (*ii*) (5 pts.) Find $\operatorname{adj}(A)$.

   (*iii*) (5 pts.) Solve, if possible, $AX = O$ and $AX = \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}^T$.

4. In each of the following questions, select the correct answer. (3 pts. each)

(a) If A and B are 3×3 symmetric matrices, then

(A) AB is symmetric
(B) AB is skew-symmetric
(C) $2A - 3B$ is symmetric
(D) None of the above

(b) Let A and B be $n \times n$ matrices, then AB is symmetric if and only if,

(A) A and B are symmetric
(B) A and B are skew-symmetric
(C) $(AB)^2 = A^2B^2$
(D) None of the above

(c) Let A be an $m \times n$ matrix. If the linear system $AX = O_{m\times 1}$ has only the trivial solution, then for any $m \times 1$ matrix B, the system $AX = B$

(A) has a unique solution
(B) has infinitely many solutions
(C) has no solution.
(D) None of the above

(d) Let A and B be $n \times n$ diagonal matrices such that $\det(A) = \det(B)$ and $\operatorname{tr}(A) = \operatorname{tr}(B)$, then

(A) $A = B$
(B) $\operatorname{tr}(A^2) = \operatorname{tr}(B^2)$
(C) $\det(AB) = \det(BA)$
(D) All of the above

(e) Let A, B and C be $n \times n$ invertible matrices. Thus,

(A) if $A(B + C) = C - A$, then $B = (A^{-1} - C^{-1} - I_n)\,C$
(B) $(AB)^T (AB)^{-1} = B^{-1}A^{-1}B^TA^T$
(C) $CA + AB = A(B + C)$
(D) All of the above

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1. (a) (*i*) (5 pts.) $A^{-1} = \begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix}$.

   (*ii*) (5 pts.) $\operatorname{tr}(A) = 1$ & $\operatorname{tr}(A^{-1}) = 1 \implies \operatorname{tr}(A) - \operatorname{tr}(A^{-1}) = 0$.

   (b) (5 pts.) Let $A = [a_{ij}], B = [b_{ij}], D = [d_{ij}] = A + cB$. Thus, $\operatorname{tr}(A + cB) = \operatorname{tr}(D) = \sum_{i=1}^{n} d_{ii} = \sum_{i=1}^{n} (a_{ii} + cb_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} cb_{ii} = \sum_{i=1}^{n} a_{ii} + c\sum_{i=1}^{n} b_{ii} = \operatorname{tr}(A) + c\operatorname{tr}(B)$.

   (c) (5+5+5 pts.) $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 5 & 3 & 1 \\ 1 & 2 & -a & a \end{array}\right] \equiv \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & -a-1 & a-2 \end{array}\right]$

   (*i*) no solution: $a = -1$ (*ii*) unique solution: $a \neq -1$ (*iii*) infinitely many solutions: Never.

2. (a) (5 pts.) (I) If A is invertible, then A^{-1} exists such that $AA^{-1} = I_n \implies \det(AA^{-1}) = \det(I_n) \implies \det(A)\det(A^{-1}) = 1 \implies \det(A) \neq 0$

   (5 pts.) (II)Let $\det(A) \neq 0$. Since $adj(A)A = \det(A) I_n \implies \left[\frac{1}{\det(A)} adj(A)\right] A = I_n \implies A^{-1} = \frac{1}{\det(A)} adj(A) \implies A$ is invertible.

   (b) (10 pts.) Let A be a nonsingular matrix and suppose that X and Y are inverses of A. Thus, $AX = I = XA$ and $AY = I = YA$. Now, $X = XI = X(AY) = (XA)Y = IY = Y$.

   (c) (10 pts.) Let $S = \frac{1}{2}(A + A^T), K = \frac{1}{2}(A - A^T)$. Clearly, $S + K = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = A$ and $S^T = \left[\frac{1}{2}(A + A^T)\right]^T = \frac{1}{2}\left(A^T + (A^T)^T\right) = S$ & $K^T = \left[\frac{1}{2}(A - A^T)\right]^T = \frac{1}{2}(A^T - A) = -K$.

3. (a) (10 pts.) $\det(\operatorname{adj}(2B)) = [\det(2B)]^2 = [2^3 \det(B)]^2 = \left[2^3\left(-\frac{1}{2}\right)\right]^2 = 16$.

   (b) (*i*) (5 pts.) $\det(A^{-1}) = \begin{vmatrix} 1 & -2 & 2 & 1 \\ 1 & 1 & -1 & 4 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 2 & 1 \\ 0 & 3 & -3 & 3 \\ 0 & -1 & 3 & 2 \\ 0 & -1 & 1 & 3 \end{vmatrix} = 3\begin{vmatrix} 1 & -1 & 1 \\ -1 & 3 & 2 \\ -1 & 1 & 3 \end{vmatrix} = 3\begin{vmatrix} 1 & -1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{vmatrix} = 24 \implies \det(A) = \frac{1}{24}$.

   (*ii*) (5 pts.) $\operatorname{adj}(A) = \det(A) A^{-1} = \frac{1}{24}\begin{bmatrix} 1 & -2 & 2 & 1 \\ 1 & 1 & -1 & 4 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 2 \end{bmatrix} = \begin{vmatrix} \frac{1}{24} & -\frac{1}{12} & \frac{1}{12} & \frac{1}{24} \\ \frac{1}{24} & \frac{1}{24} & -\frac{1}{24} & \frac{1}{6} \\ -\frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} \\ -\frac{1}{24} & \frac{1}{24} & -\frac{1}{24} & \frac{1}{12} \end{vmatrix}$.

   (*iii*) (1 point) The solution of $AX = O$ is the trivial solution since A is invertible. [i.e., $x_1 = x_2 = x_3 = x_4 = 0$].

   (4 pts.) The solution of $AX = \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}^T$ is $X = A^{-1}\begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}^T \implies x_1 = 0, x_2 = -3, x_3 = -2, x_4 = -3$.

4. (a) If A and B are 3×3 symmetric matrices, then

   (C) $2A - 3B$ is symmetric

   (b) Let A and B be $n \times n$ matrices, then AB is symmetric if and only if,

   (D) None of the above

   (c) Let A be an $m \times n$ matrix. If the linear system $AX = O_{m\times 1}$ has only the trivial solution, then for any $m \times 1$ matrix B, the system $AX = B$

   (D) None of the above

   (d) Let A and B be $n \times n$ diagonal matrices such that $\det(A) = \det(B)$ and $\operatorname{tr}(A) = \operatorname{tr}(B)$, then

   (C) $\det(AB) = \det(BA)$

   (e) Let A, B and C be $n \times n$ invertible matrices. Thus,

   (A) if $A(B + C) = C - A$, then $B = (A^{-1} - C^{-1} - I_n)\, C$