Kuwait University | ***Math111. Linear Algebra*** | **Date: May 15, 2018**
Department of Mathematics | ***Final Exam*** | **Duration: Two Hours**

Calculators and communication devices are not allowed
Answer the following questions

1. (a) (10 pts.) Find for what values of a and b the system

$$\begin{array}{rcrcrcl} ax & & & + & bz & = & 2 \\ ax & + & ay & + & 4z & = & 4 \\ & & ay & + & 2z & = & b \end{array}$$

has:
(i) a unique solution (ii) a one-parameter solution

(iii) a two-parameter solution (iv) no solution

(b) (5+5 pts.) Let A be an $n \times n$ matrix and k a scalar. Show that:

(i) $\det(kA) = k^n \det(A)$.

(ii) $\operatorname{tr}(kA) = k \operatorname{tr}(A)$.

2. (a) (10 pts.) Let U and V be linearly independent vectors in $\mathbb{R}^3$. Show that U, V, and $U \times V$ form a basis for $\mathbb{R}^3$.

(b) Let $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

(i) (5 pts.) Find B^{-1}.

(ii) (5 pts.) Find $\operatorname{adj}(B)$.

3. Let $A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \end{bmatrix}$.

(a) (2 pts.) Find the reduced row echelon form of A.

(b) (2 pts.) Find a basis for the column space of A.

(c) (2 pts.) Find a basis for the row space of A.

(d) (2 pts.) Find a basis for the null space of A.

(e) (2 pts.) Find nullity(A) and nullity$\left(A^T\right)$.

4. (a) (10 pts.) Show that if A and B are similar matrices, then A^3 and B^3 are similar.

(b) (10 pts.) Determine whether $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$, is diagonalizable.

5. (a) (10 pts.) Show that, if X and Y are orthogonal vectors in $\mathbb{R}^n$, then

$$\|X+Y\|^2 = \|X\|^2 + \|Y\|^2.$$

(b) Let π be the plane which contains the point $P(1, 2, -1)$ and the line

$L : \dfrac{x+1}{2} = \dfrac{y-1}{-1} = \dfrac{z-2}{1}$.

(i) (5 pts.) Find a vector N which is normal to π.

6. In each of the following questions, select the correct answer. (2 pts. each)

(a) Let A be a 3×3 matrix, such that $AX = -X, AY = Y, AZ = -2Z$, for nonzero vectors X, Y and Z in $\mathbb{R}^3$. Then

(A) $\det(A) = 2$
(B) the characteristic equation of A is: $\lambda^3 + 2\lambda^2 - \lambda - 2 = 0$
(C) all of the above
(D) none of the above

(b) Let A be a nonzero 4×7 matrix, then A may be

(A) of rank 5
(B) of nullity 7
(C) row equivalent to a 7×4 matrix
(D) none of the above

(c) Let A be an $n \times n$ matrix in a reduced row echelon form. Then

(A) $A = I_n$
(B) $A^2 = A$
(C) all of the above
(D) none of the above

(d) For all vectors U, V and W in $\mathbb{R}^n$

(A) $\|U + V + W\| \leq \|U\| + \|V\| + \|W\|$
(B) $\|U + V + W\| = \|U\| + \|V\| + \|W\|$
(C) $\|U + V + W\| \geq \|U\| + \|V\| + \|W\|$
(D) none of the above

(e) Let V be a vector space and W_1 and W_2 be subspaces of V. If $B_1 = \{X, Y\}$ and $B_2 = \{X, Z\}$ are bases for W_1 and W_2, respectively, then

(A) $\dim(V) \geq 4$
(B) $\dim(W_1 \cap W_2) = 1$
(C) all of the above
(D) none of the above

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1. (a) (10 pts.) The augmented matrix of the linear system is:

$$\left[\begin{array}{ccc|c} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{array}\right] \equiv \left[\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & a & 2 & b \end{array}\right] \equiv \left[\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & 0 & b-2 & b-2 \end{array}\right]$$

   (i) a unique solution: if $a \neq 0$ and $b \neq 2$.
   (ii) a one-parameter solution: if $a \neq 0$ and $b = 2$
   (iii) a two-parameter solution: if $a = 0$ and $b = 2$.
   (iv) no solution: if $a = 0$ and $\{b = 0 \text{ or } b = 4\}$

   (b) (5+5 pts.) Let $A = [a_{ij}]$ be an $n \times n$ matrix and k a scalar. Let $kA = B = [b_{ij}]$, then $b_{ij} = ka_{ij}$ for $1 \le i \le n, 1 \le j \le n$.

   (i) $\det(kA) = \det(B) = \sum(\pm)\ b_{1j_1}b_{2j_2}...b_{nj_n} = \sum(\pm)\ (ka_{1j_1})(ka_{2j_2})...(ka_{nj_n})$
   $= \sum(\pm)\ k^n(a_{1j_1}a_{2j_2}...a_{nj_n}) = k^n\sum(\pm)\ a_{1j_1}a_{2j_2}...a_{nj_n} = k^n \det(A)$.

   (ii) $\operatorname{tr}(kA) = \operatorname{tr}(B) = \sum_{i=1}^{n} b_{ii} = \sum_{i=1}^{n}(ka_{ii}) = k\sum_{i=1}^{n} a_{ii} = k \operatorname{tr}(A)$.

2. (a) Let $c_1, c_2, c_3 \in \mathbb{R}$ be scalars such that $O = c_1U + c_2V + c_3(U \times V) \implies$
$(U \times V)\cdot O = 0 = (U \times V)\cdot(c_1U + c_2V + c_3U \times V)$
$= c_1(U \times V)\cdot U + c_2(U \times V)\cdot V + c_3(U \times V)\cdot(U \times V) = 0 + 0 + c_3\|U \times V\|^2$.
Since U and V are linearly independent, then they are not parallel and hence $\|U \times V\|^2 \neq 0$. Then, $c_3 = 0$. Thus, $c_1U + c_2V = O$. Since U and V are linearly independent, then $c_1 = c_2 = 0$. Therefore, $c_1 = c_2 = c_3 = 0$. Thus, the vectors U, V and $U \times V$ are linearly independent in $\mathbb{R}^3$. Therefore, they form a basis for $\mathbb{R}^3$.

   (b) (i) (5+5 pts.) $B^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$, (ii) $\det(B) = -1, \implies$

$$\operatorname{adj}(B) = |B|\ B^{-1} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}.$$

3. (a) $A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \end{bmatrix} \equiv ... \equiv \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, which is the RREF of A.

   (b) Basis for the column space of $A = \left\{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}\right\}$.

   (c) Basis for the row space of $A = \left\{\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}\right\}$.

   (d) Basis for the null space of $A = \left\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}\right\}$.

   (e) $\text{nullity}(A) = 2$ and $\text{nullity}(A^T) = 1$.

4. (a) $A = X^{-1}BX \implies A^3 = (X^{-1}BX)^3 = X^{-1}B\underline{XX^{-1}}B\underline{XX^{-1}}BX = X^{-1}B^3X \implies$

(b) $|\lambda I_3 - A| = \begin{vmatrix} \lambda+1 & 0 & 0 \\ 0 & \lambda-2 & 1 \\ -1 & 0 & \lambda-2 \end{vmatrix} = (\lambda+1)(\lambda-2)^2.$

Let $\lambda_1 = \lambda_2 = 2$, $\lambda_3 = -1$ be the eigenvalues of A.

Solve: $(2I_3 - A) X = O$, $\left[\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right] \equiv \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$

The eigenvectors corresponding to the eigenvalues $\lambda_1 = \lambda_2 = 2$, are $X = r \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $r \neq 0$.

The algebraic multiplicity, of $\lambda = 2$ is 2, But the geometric multiplicity is 1. Therefore, A is not diagonalizable.

5. (a) Since X and Y are orthogonal, then $X \cdot Y = 0$. $\|X+Y\|^2 = (X+Y)\cdot(X+Y) = X \cdot X + 2X \cdot Y + Y \cdot Y = \|X\|^2 + 0 + \|Y\|^2 = \|X\|^2 + \|Y\|^2$.

(b) (*i*) Consider $P_1(-1,1,2) \in L$. Let $V = \overrightarrow{P_1P} = (2,1,-3)$. Thus,

$$N = |(2,1,-3) \times (2,-1,1)| = \begin{vmatrix} i & j & k \\ 2 & 1 & -3 \\ 2 & -1 & 1 \end{vmatrix} = (-2,-8,-4).$$

(*ii*) An equation for π is: $-2(x+1) - 8(y-1) - 4(z-2) = 0$ or $x + 4y + 2z - 7 = 0$.

6. (a) Let A be a 3×3 matrix, such that $AX = -X$, $AY = Y$, $AZ = -2Z$, for nonzero vectors X, Y and Z in $\mathbb{R}^3$. Then

(C) all of the above

(b) Let A be a nonzero 4×7 matrix, then A may be

(D) none of the above

(c) Let A be an $n \times n$ matrix in the reduced row echelon form. Then

(D) none of the above

(d) For all vectors U, V and W in $\mathbb{R}^n$

(A) $\|U + V + W\| \leq \|U\| + \|V\| + \|W\|$

(e) Let V be a vector space and W_1 and W_2 be subspaces of V. If $B_1 = \{X, Y\}$ and $B_2 = \{X, Z\}$ are bases for W_1 and W_2, respectively, then

(D) none of the above