Kuwait University	*Math111. Linear Algebra*	**Date: Dec. 28, 2017**
Department of Mathematics	*Final Exam*	**Duration: Two Hours**

Calculators and communication devices are not allowed

Answer the following questions

1. (5 pts.) Find parametric equations of the line through the points $A(1,2,3)$ and $B(2,1,4)$.

2. (5 pts.) Show that the characteristic polynomials of a square matrix A and its transpose, A^T, are the same.

3. (5 pts.) Assuming that the stated inverses exist for a nonsingular matrix A, show that $(I+AB)^{-1}A = A(I+BA)^{-1}$.

4. (10 pts.) Show that
$$\begin{vmatrix} c_1+2a_1 & b_1 & a_1 \\ c_3+2a_3 & b_3 & a_3 \\ c_2+2a_2 & b_2 & a_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

5. Let $W = \text{span}\{(1,1,0),(1,0,1),(2,1,1),(0,1,-1)\}$.

   (a) (7 pts.) Is $(9,0,9)$ in W? Justify your answer.

   (b) (3 pts.) Is $W = \mathbb{R}^3$? Justify your answer.

6. (5 pts.) Show that if $S = \{V_1, V_2, ..., V_k\}$ is a linearly dependent set of vectors in a vector space V, then one of the vectors in S is a linear combination of the other vectors.

7. (a) (5 pts.) Determine whether $W = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2=z^2\}$ is a subspace of $\mathbb{R}^3$.

   (b) (5 pts.) Let A, B, and C be $n \times n$ matrices. Show that if A is similar to B, and B is similar to C, then A is similar to C.

   (c) (5 pts.) Show that similar matrices have equal determinants.

8. Let $A(1,2,3), B(2,1,4)$, and $C(3,1,1)$ be points in $\mathbb{R}^3$.

   (a) (10 pts.) Find an equation of the plane through A, B and C.

   (b) (5 pts.) Find the area of the triangle ABC.

9. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

   (a) (5 pts.) Find the eigenvalues of A.

   (b) (5 pts.) Find a basis for every eigenspace of A.

   (c) (5 pts.) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

10. In each of the following questions, select the correct answer. (3 pts. each)

(I) Let A be a 6×4 matrix, then the

(A) rows of A are linearly independent
(B) rows of A are linearly dependent
(C) columns of A are linearly independent
(D) columns of A are linearly dependent

(II) Let A and B be similar matrices, then

(A) rank (A) = rank (B)
(B) nullity (A) = nullity (B)
(C) the eigenvalues of A and B are the same
(D) all of the above

(III) If A and B are row equivalent matrices, then

(A) $\det(A) = \det(B)$
(B) the column spaces of A and B are the same
(C) the row spaces of A and B are the same
(D) all of the above

(IV) Let A and B be 3×3 matrices with $\det(A) = 2$ and $\det(B) = 3$. Then the $\det(2AB^{-1})$ is equal to

(A) 4/3
(B) 8/3
(C) 16/3
(D) 3/16

(V) Let A and B be $n \times n$ invertible matrices. If $\det(A) = \det(B)$, then

(A) $A = B$.
(B) A and B have the same eigenvalues.
(C) A and B are similar matrices.
(D) none of the above.

1. $U = \overrightarrow{AB} = (2-1, 1-2, 4-3) = (1,-1,1) \implies$
   Parametric equations are: $x = 1+t, y = 2-t, z = 3+t$.

2. The Characteristic Polynomial A is $|\lambda I_n - A| = \left|(\lambda I_n - A)^T\right| = \left|(\lambda I_n)^T - A^T\right| = \left|\lambda I_n - A^T\right|$, which is the Characteristic Polynomial of A^T.

3. Solution (1): $(I+AB)^{-1}A = A(I+BA)^{-1} \iff [(I+AB)^{-1}A]^{-1} = [A(I+BA)^{-1}]^{-1} \iff$ $A^{-1}(I+AB) = (I+BA)A^{-1} \iff A^{-1} + A^{-1}(AB) = A^{-1} + (BA)A^{-1} \iff$
   $A^{-1} + IB = A^{-1} + BI$.
   Solution (2): {Actually, this is true for any matrix A, even when A is singular}.
   $(I+AB)^{-1}A = A(I+BA)^{-1} \iff A = (I+AB)A(I+BA)^{-1} \iff$
   $A = (A+ABA)(I+BA)^{-1} \iff A = A(I+BA)(I+BA)^{-1} \iff A = AI = A$.

4. $\begin{vmatrix} c_1+2a_1 & b_1 & a_1 \\ c_3+2a_3 & b_3 & a_3 \\ c_2+2a_2 & b_2 & a_2 \end{vmatrix} = \begin{vmatrix} c_1 & b_1 & a_1 \\ c_3 & b_3 & a_3 \\ c_2 & b_2 & a_2 \end{vmatrix} = -\begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

5. (a) $c_1(1,1,0)+c_2(1,0,1)+c_3(2,1,1)+c_4(0,1,-1) = (9,0,9) \implies \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 9 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 9 \end{array}\right] \equiv$
   $... \equiv \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \implies c_1 = -r-s, c_2 = -r+s+9, c_3 = r, c_4 = s$.
   Yes, $(9,0,9) \in W$.$\{(-r-s)(1,1,0) + (-r+s+9)(1,0,1) + r(2,1,1) + s(0,1,-1) = (9,0,9)\}$.
   (b) No, $W \neq \mathbb{R}^3$. A basis for W is $\{_1(1,1,0),(1,0,1)\} \implies \dim(W) = 2$.

6. $S = \{V_1, V_2, ..., V_k\}$ is linearly dependent $\implies \exists\, c_1, c_2, ..., c_n \in \mathbb{R}$ not all are zeros such that $O = c_1V_1 + c_2V_2 + ... + c_kV_k$. Let $c_j \neq 0$, then $-c_jV_j = c_1V_1 + c_2V_2 + ... + c_{j-1}V_{j-1} + c_{j+1}V_{j+1} + ... + c_kV_k \implies V_j = \left(\frac{c_1}{-c_j}\right)V_1 + \left(\frac{c_2}{-c_j}\right)V_2 + ... + \left(\frac{c_{j-1}}{-c_j}\right)V_{j-1} + \left(\frac{c_{j+1}}{-c_j}\right)V_{j+1} + ... + \left(\frac{c_k}{-c_j}\right)V_k$.

7. (a) $W = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = z^2\}$ is not a subspace of $\mathbb{R}^3$. Since, $(0,2,2),(1,0,1) \in W$, but $(0,2,2)+(1,0,1) = (1,2,3) \notin W$.
   (b) A is similar to B, and B is similar to $C \implies \exists$ non singular matrices X and Y such that $A = X^{-1}BX$ and $B = Y^{-1}CY \implies A = X^{-1}(Y^{-1}CY)X = (X^{-1}Y^{-1})C(YX) = Z^{-1}CZ$, where $Z = YX$.
   (c) Let A and B be similar matrices $\implies \exists$ non singular matrix X such that $A = X^{-1}BX \implies |A| = |X^{-1}BX| = |X^{-1}||B||X| = \frac{1}{|X|}|B||X| = |B|$.

8. (5 pts.) $\overrightarrow{AB} = (1,-1,1), \overrightarrow{BC} = (1,0,-3)$. Let $N = \overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 1 & 0 & -3 \end{vmatrix} = (3,4,1)$.

   (a) (5 pts.) Equation of the plane: $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \implies$
   $3(x-1) + 4(y-2) + (1)(z-3) = 0 \implies \boxed{3x+4y+z-14=0}$.

9. (a) $|\lambda I_3 - A| = \begin{vmatrix} \lambda-2 & 0 & 0 \\ 0 & \lambda-1 & -1 \\ 0 & -1 & \lambda-1 \end{vmatrix} = (\lambda-2)\left[(\lambda-1)^2-1\right] = (\lambda-2)\left(\lambda^2-2\lambda\right) =$ $\lambda(\lambda-2)^2$. Let $\lambda_1=\lambda_2=2, \lambda_3=0$.

(b) Solve: $((2)\,I_3 - A)\,X = O$, $\left[\begin{array}{ccc|c} 2-2 & 0 & 0 & 0 \\ 0 & 2-1 & -1 & 0 \\ 0 & -1 & 2-1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \equiv$

$\ldots \equiv \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow$

The eigenvectors corresponding to the eigenvalue 2, are $X = r\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \neq O$, i.e., $(r^2+s^2 \neq 0)$

Solve: $((0)\,I_3 - A)\,X = O$, $\left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \equiv \ldots \equiv \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow$

The eigenvectors corresponding to the eigenvalue 0, are $X = r\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \neq O$, i.e., $r \neq 0$.

(c) Take $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$. Then $P^{-1}AP = D$.

10. (a) Let A be a 6×4 matrix, then the

rows of A are linearly dependent

(b) Let A and B be similar matrices, then

all of the above

(c) If A and B are row equivalent matrices, then

the row spaces of A and B are the same

(d) Let A and B be 3×3 matrices with $\det(A) = 2$ and $\det(B) = 3$.

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(e) Let A and B be $n \times n$ invertible matrices. If $\det(A) = \det(B)$, then

none of the above